International Journal of Mathematical Sciences and Optimization: Theory and Applications
11(1), 2025, Pages 107 - 113
https://doi.org/10.5281/zenodo.15176079

Perfect Product of two Squares in Finite Full Transformation Semigroup

A.T Imam [1*], M. Balarabe [1], S. Kasim [1], C. Eze [1]

1. Department of Mathematics, Faculty of Physical Sciences, Ahmadu Bello University, Zaria, Nigeria.
* Corresponding author: atimam@abu.edu.ng*, abusaeed.musa@gmail.com, sagirkasim@gmail.com, ezewisdom8@gmail.com

Article Info
Received: 24 January 2025 Revised: 17 March 2025
Accepted: 01 April 2025 Available online: 10 April 2025

Abstract

In this paper, we investigate the concept of the perfect product of two squares in the context of finite full transformation semigroups. We provide a comprehensive analysis of the conditions under which the product of two idempotent elements in a transformation semigroup forms a perfect product of two squares. Specifically, we examine the relationship between the kernel and image of idempotents, as well as the interplay between the domain and image of these transformations. The main result establishes that for two idempotent elements α and β in T_n, if the domain and image of α and β satisfy certain equivalence conditions, then their product is a perfect product of two squares. We also explore related properties of disjoint cycles and how these contribute to the structural characteristics of the semigroup. Our findings extend the existing theory of transformation semigroups and offer valuable insights into the decomposition of semigroup elements into squares, contributing to the broader field of semigroup theory.

Keywords: Perfect Square, Perfect Product, Permutation, Full transformation Semigroup.
MSC2010: 06F05.

1 Introduction

The study of the semigroup of transformations $S(X)$ and the full transformation semigroup $T(X)$ has captivated mathematicians for decades, forming the backbone of various mathematical disciplines. The semigroup $S(X)$ represents permutations of elements within a set, while $T(X)$ encompasses all mappings from a finite set X to itself, with the semigroup operation defined as the composition of mappings. These mathematical structures are pivotal in understanding the interplay between algebraic operations and combinatorial structures. One intriguing aspect of $T(X)$ lies in the investigation of squares and their products. An element $\alpha \in T(X)$ is termed a square if there exists an element $\beta \in T(X)$ such that $\alpha = \beta^2$. The product of squares in $T(X)$ refers to the product of two elements, each of which is a square within the semigroup. Formally, given two square elements α and γ in $T(X)$, their product is denoted as $\alpha\gamma$. The notion of square roots in semigroups, specifically within $T(X)$, adds another layer of depth. A square root of an element $\alpha \in T(X)$ is an element $\beta \in T(X)$ such that $\alpha = \beta^2$. For α and β in $T(X)$, where their respective square roots are δ and γ, the product of squares is represented as $(\delta\gamma)^2$. This concept has garnered significant attention due to its potential applications in group theory, lattice theory, and universal algebra. It is important to note that in this work, the composition of transformations is performed from left to right. Moreover, composition of transformations in $T(X)$ is not commutative; that is, for transformations $\alpha, \beta \in T(X)$, in general, $\alpha\beta \neq \beta\alpha$. The study of squares and their products in $T(X)$ is a relatively recent area of research. Foundational works by Snowden [1], Higgins [2], [3], and Annin [4] have laid the groundwork by exploring the existence and properties of square roots within finite full transformation semigroups. The seminal contributions by Howie [8], [7] provide crucial insights into the multiplication of elements in transformation semigroups, which serve as a cornerstone for understanding the product of squares. Recent developments in transformation semigroup theory have led to the study of quasi-idempotents and their role in semigroup generation. In particular, [9] investigated quasi-idempotent elements within the semigroup of partial order-preserving transformations. They demonstrated that the semigroup PO_n of all partial order-preserving transformations on a finite set X_n is quasi-idempotent generated. Furthermore, they established an upper bound for the quasi-idempotent rank of PO_n, given by $\lceil \frac{5n-4}{2} \rceil$. This result provides new insights into the structural composition of transformation semigroups. Additionally, [10] examined the collapse of order-preserving and idempotent elements in the semigroup of order-preserving full contraction transformations. Their study explores the number of collapsible elements in this semigroup, providing a formula for their enumeration and shedding light on the behavior of idempotent elements in transformation semigroups. In this paper, we investigate the perfect product of two squares within the finite full transformation semigroup $T(X)$. Our primary focus is to characterize and analyze the algebraic properties of such products, providing a deeper understanding of their structure and behavior. By incorporating results from quasi-idempotent and idempotent order-preserving transformations, we extend the scope of previous studies to include their interactions within transformation semigroups.

2 Preliminaries

Definition 2.1. A semigroup is a set S_ϱ together with an associative binary operation usually denoted by juxtaposition ie $(xy)z = x(yz)$ $\forall x, y, z \in S_\varrho$. The semigroup S_ϱ is called monoid if it has identity that is if it contains an element 1 with the property $x.1 = 1.x = x$ $\forall x \in S_\varrho$. An elment $0 \in S_\varrho$ with the property that $0x = x0 = 0$ $\forall x \in S_\varrho$.is called zero element of S_ϱ and S_ϱ a semigroup with zero. If S_ϱ has no identity we can adjoin an extra identity or zero to S_ϱ, so that S becomes a monoid or semigroup with zero. [6]

We write S^1_ϱ and S^0_ϱ to denote the semigroup with identity or zero adjoined if necessary. Thus,

$$S^1_\varrho = \begin{cases} S_\varrho, & \text{if } S_\varrho \text{ has an identity;} \\ S_\varrho \cup \{1\}, & \text{otherwise.} \end{cases}$$

and

$$S^0_\varrho = \begin{cases} S_\varrho, & \text{if } S_\varrho \text{ has a zero;} \\ S_\varrho \cup \{0\}, & \text{otherwise,} \end{cases}$$

where we define $1s = s1 = S_\varrho$, $11 = 1$ and $0s = s_\varrho 0 = 00 = 0$ $\forall s \in S_\varrho$. [6]

Definition 2.2. *[5]*A non-empty subset T of a semigroup S_ϱ is called a subsemigroup of S_ϱ, if it is closed under the binary operation of S_ϱ i.e. for all x, y in T, xy is in Tor $T^2 \subseteq T$. For example, if S_ϱ is a semigroup with zero, the set T of all elements of S_ϱ, in which the product of any two elements is zero, is a subsemigroup of S_ϱ.

International Journal of Mathematical Sciences and Optimization: Theory and Applications
11(1), 2025, Pages 107 - 113
https://doi.org/10.5281/zenodo.15176079

Definition 2.3. [5]Let $X_n = \{1, 2, ..., n\}$. A mapping α:dom$(\alpha) \subset X_n \to Im(\alpha)$. X_n is called full transformation of X_n if $dom(\alpha) = X_n$. The set of all full transformations of X_n, forms a semgroup under composition of mappings called the full transformation semigroup.

Definition 2.4. An element e of a semigroup S_ϱ is called **idempotent** if $e \cdot e = e$, where $\cdot$ denotes the binary operation in S_ϱ. That is, an element is idempotent if, when composed with itself, it yields itself.

Definition 2.5. Let S_ϱ be a semigroup. An element $s \in S_\varrho$ is a perfect product of two squares if there exist elements $\alpha, \beta \in S_\varrho$ such that

$$s = \alpha^2 \cdot \beta^2,$$

where $\alpha^2 = \alpha \cdot \alpha$ and $\beta^2 = \beta \cdot \beta$ are squares of elements in S_ϱ.

Definition 2.6. Let X be a finite set, and let $T(X)$ denote the full transformation semigroup on X. An element $\alpha \in T(X)$ is called a **square** if there exists an element $\delta \in T(X)$ such that $\alpha = \delta^2$. The product of two squares in $T(X)$ is defined as the element $\alpha\beta$, where both α and β are squares in $T(X)$.

3 Main Results

Definition 3.1. Let α and β be elements in $T(X)$ with square roots δ and γ in $T(X)$ respectively, then $\alpha\beta$ are said to be perfect product of two square if $\alpha\beta = (\delta\gamma)^2$.

Example 3.2.

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 45 & 6 & 7 \\ 3 & 4 & 6 & 7 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 45 & 6 & 7 \\ 6 & 7 & 1 & 2 & 3 & 4 \end{pmatrix}^2 \Rightarrow \delta = \begin{pmatrix} 1 & 2 & 3 & 45 & 6 & 7 \\ 6 & 7 & 1 & 2 & 3 & 4 \end{pmatrix}$$
$$\beta = \begin{pmatrix} 1 & 2 & 3 & 45 & 6 & 7 \\ 3 & 5 & 6 & 7 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 45 & 6 & 7 \\ 6 & 7 & 1 & 2 & 3 & 5 \end{pmatrix}^2 \Rightarrow \gamma = \begin{pmatrix} 1 & 2 & 3 & 45 & 6 & 7 \\ 6 & 7 & 1 & 2 & 3 & 5 \end{pmatrix}$$
$$\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 45 & 6 & 7 \\ 6 & 7 & 1 & 2 & 3 & 5 \end{pmatrix}$$
$$\delta\gamma = \begin{pmatrix} 1 & 2 & 3 & 45 & 6 & 7 \\ 3 & 5 & 6 & 7 & 1 & 2 \end{pmatrix}$$
$$(\delta\gamma)^2 = \begin{pmatrix} 1 & 2 & 3 & 45 & 6 & 7 \\ 6 & 7 & 1 & 2 & 3 & 5 \end{pmatrix}$$
$$\Rightarrow \alpha\beta = (\delta\gamma)^2.$$

Hence, $\alpha\beta$ is a perfect product of two squares.

Theorem 3.3. Let α, β be any two idempotents of height $r \geq 2$ in T_n. Then $\alpha\beta$ is a perfect product of two squares if and only if $dom\alpha = dom\beta$ or $im\alpha = im\beta$.

Proof. Suppose that $dom\alpha = dom\beta$ or $im\alpha = im\beta$. We will show that $\alpha\beta = (\delta\gamma)^2$. If $dom\alpha = dom\beta$, then $x\alpha = x\beta$ for all $x \in X_n \setminus$kernels. Similarly, if $im\alpha = im\beta$, then $x\alpha = x\beta$ for all $x \in X_n$. Since α and β are both idempotent, clearly they are squares. Suppose δ and γ are the square roots of α and β, respectively, then $\alpha = \delta^2$ and $\beta = \gamma^2$. Hence, $\alpha\beta = \delta^2\gamma^2$. But since $x\alpha = x\beta$, we have:

$$\alpha\beta = (\delta\gamma)^2.$$

Conversely, suppose $\alpha\beta = (\delta\gamma)^2$. We will show that $dom\alpha = dom\beta$ or $im\alpha = im\beta$. Since $\alpha\beta = (\delta\gamma)^2$, clearly:

$$\alpha\beta = (\delta\gamma)^2 = \delta^2\gamma^2,$$

which implies that $\alpha = \delta^2$ and $\beta = \gamma^2$. Thus, for all $x \in X_n$:

$$x\alpha = x\beta,$$

which implies that $im\alpha = im\beta$ or $dom\alpha = dom\beta$. $\square$

IJMSO

INTERNATIONAL JOURNAL OF MATHEMATICAL SCIENCES AND OPTIMIZATION: THEORY AND APPLICATIONS
11(1), 2025, PAGES 107 - 113
HTTPS://DOI.ORG/10.5281/ZENODO.15176079

Example 3.4. $\alpha = \begin{pmatrix} 1 & 23 & 45 & 6 \\ 1 & 2 & 4 & 6 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 23 & 45 & 6 \\ 1 & 3 & 5 & 6 \end{pmatrix}$

$\alpha\beta = \begin{pmatrix} 1 & 23 & 45 & 6 \\ 1 & 2 & 4 & 6 \end{pmatrix}\begin{pmatrix} 1 & 23 & 45 & 6 \\ 1 & 3 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 23 & 45 & 6 \\ 1 & 3 & 5 & 6 \end{pmatrix} = \delta^2\gamma^2 = (\delta\gamma)^2$

Example 3.5. $\alpha = \begin{pmatrix} 1 & 23 & 45 & 67 & 8 \\ 1 & 3 & 4 & 6 & 8 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 23 & 4 & 56 & 78 \\ 1 & 3 & 4 & 6 & 8 \end{pmatrix}$

$\alpha\beta = \begin{pmatrix} 1 & 23 & 45 & 67 & 8 \\ 1 & 3 & 4 & 6 & 8 \end{pmatrix}\begin{pmatrix} 1 & 23 & 4 & 56 & 78 \\ 1 & 3 & 4 & 6 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 23 & 45 & 67 & 8 \\ 1 & 3 & 4 & 6 & 8 \end{pmatrix} = \delta^2\gamma^2 = (\delta\gamma)^2$

Example 3.6. this example is a counter example
$\alpha = \begin{pmatrix} 1 & 234 & 56 \\ 1 & 2 & 5 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 2 & 3 & 45 & 6 \\ 1 & 2 & 3 & 4 & 6 \end{pmatrix}$

$\alpha\beta = \begin{pmatrix} 1 & 234 & 56 \\ 1 & 2 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 45 & 6 \\ 1 & 2 & 3 & 4 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 234 & 56 \\ 1 & 2 & 4 \end{pmatrix}$

clearly $\alpha\beta$ is not an idempotent. and element of this nature may or may not be a square.

Theorem 3.7. Let $\alpha, \beta \in J_n(T_n)$ each of which is a square, then $\alpha\beta$ is a perfect product of two squares if and only if α and β are disjoint.

Proof. Suppose α and β are disjoint. Then $\alpha\beta = \beta\alpha$. Since α and β are squares, there exist δ and γ in J_n such that $\alpha = \delta^2$ and $\beta = \gamma^2$. Thus, $\alpha\beta = \delta^2\gamma^2$, and therefore, δ^2 and γ^2 can be written as the product of disjoint cycles. We have:

$$\delta^2 = (c_1, c_2, \ldots, c_n)^2 \quad \text{and} \quad \gamma^2 = (d_1, d_2, \ldots, d_n)^2,$$

where for $i = 1, 2, \ldots, n$, c_i and d_i are cycles. Now, we have:

$$\delta^2 = (c_1^2, c_2^2, \ldots, c_n^2),$$

and

$$\gamma^2 = (d_1^2, d_2^2, \ldots, d_n^2).$$

Thus:

$$\delta^2\gamma^2 \Leftrightarrow (c_1^2, c_2^2, \ldots, c_n^2)(d_1^2, d_2^2, \ldots, d_n^2) \Leftrightarrow (c_1, c_2, \ldots, c_n)^2(d_1, d_2, \ldots, d_n)^2 \Leftrightarrow [(c_1, c_2, \ldots, c_n)(d_1, d_2, \ldots, d_n)]^2.$$

Hence:

$$\Leftrightarrow (c_1d_1, c_2d_2, \ldots, c_nd_n)^2 = (\delta\gamma)^2.$$

Therefore:

$$\Leftrightarrow \alpha\beta = (\delta\gamma)^2.$$

□

Example 3.8. $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 4 & 5 & 6 \end{pmatrix}^2 \Rightarrow$ *the square root*

$\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 4 & 5 & 6 \end{pmatrix}$

$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 5 & 6 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 6 & 4 & 5 \end{pmatrix}^2 \Rightarrow$ *the squre root*

$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 6 & 4 & 5 \end{pmatrix}$

$\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix}$

$\delta\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 6 & 4 & 5 \end{pmatrix}$

$(\delta\gamma)^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix} \Rightarrow \alpha\beta = (\delta\gamma)^2$

Hence $\alpha\beta$ is a perfect product of two square. It is possible in group of permutation if Shift(α) = Fix(β) and Shift(β) = Fix(α) which makes it disjoint.

Theorem 3.9. Suppose that α, $\beta \in J_{n-1}(T_n)$, $\ker\alpha = \ker\beta = |i, j|$, $\alpha = \delta^2$, and $\beta = \gamma^2$. Then,

$$\alpha\beta = (\delta\gamma)^2 \quad \text{if and only if} \quad \mathrm{im}\alpha = X_n \setminus \{i\}, \quad \mathrm{im}\beta = X_n \setminus \{j\},$$

and

$$\alpha \mid \mathrm{im}(\alpha) \setminus \{j\} = \beta \mid \mathrm{im}(\beta) \setminus \{i\}.$$

Proof. Assume $\alpha\beta = (\delta\gamma)^2$. We need to show (i), (ii), and (iii).

(i) $\mathbf{im}(\alpha) = X_n \setminus \{i\}$: Assume, by contradiction, that there exists $x \in X_n$ such that $x\alpha = i$. Then:

$$x(\alpha\beta) = (x\alpha)\beta = i\beta.$$

Since $\alpha\beta = (\delta\gamma)^2$, it implies:

$$x(\alpha\beta) = x(\delta\gamma)^2 \Rightarrow (x\alpha)\beta = x(\delta\gamma)^2 \Rightarrow i\beta = (\delta\gamma)^2,$$

which leads to a contradiction.

(ii) $\mathbf{im}(\beta) = X_n \setminus \{j\}$: Similarly, assume for the sake of contradiction that there exists $x \in X_n$ such that $x\beta = j$. Using $\alpha\beta = (\delta\gamma)^2$, one can show that $j\alpha = (\delta\gamma)^2$, which again leads to a contradiction.

(iii) $\alpha \mid \mathrm{im}(\alpha) \setminus \{j\} = \beta \mid \mathrm{im}(\beta) \setminus \{i\}$:
Suppose there exists $x \in \mathrm{im}(\alpha) \setminus \{j\}$ such that $x\alpha \neq x\beta$. Then, $\alpha\beta \neq (\delta\gamma)^2$, contradicting our assumption that $\alpha\beta = (\delta\gamma)^2$. Hence:

$$\alpha \mid \mathrm{im}(\alpha) \setminus \{j\} = \beta \mid \mathrm{im}(\beta) \setminus \{i\}.$$

Conversely:
Assume (i), (ii), and (iii) hold. We need to show that $\alpha\beta = (\delta\gamma)^2$.
Consider any $x \in X_n$:

$$\begin{aligned} x(\alpha\beta) &= (x\alpha)\beta = x\beta \quad \text{if } x = j, \\ x(\alpha\beta) &= (x\alpha)\beta = x\alpha \quad \text{if } x = i, \\ x(\alpha\beta) &= (x\alpha)\beta = x(\delta\gamma)^2 \quad \text{if } x \notin \{i, j\}. \end{aligned}$$

Therefore, $\alpha\beta = (\delta\gamma)^2$, and the reverse direction is established. ∎

Example 3.10. $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 89 & 10 & 11 & 12 & 13 \\ 3 & 4 & 10 & 11 & 6 & 5 & 8 & 7 & 12 & 13 & 1 & 2 \end{pmatrix} =$

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 89 & 10 & 11 & 12 & 13 \\ 2 & 3 & 4 & 10 & 7 & 8 & 6 & 5 & 11 & 12 & 13 & 1 \end{pmatrix}^2 \Rightarrow \delta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 89 & 10 & 11 & 12 & 13 \\ 2 & 3 & 4 & 10 & 7 & 8 & 6 & 5 & 11 & 12 & 13 & 1 \end{pmatrix}$

$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 89 & 10 & 11 & 12 & 13 \\ 3 & 4 & 10 & 11 & 6 & 5 & 9 & 7 & 12 & 13 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 89 & 10 & 11 & 12 & 13 \\ 2 & 3 & 4 & 10 & 7 & 9 & 6 & 5 & 11 & 12 & 13 & 1 \end{pmatrix}^2 \Rightarrow$

$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 89 & 10 & 11 & 12 & 13 \\ 2 & 3 & 4 & 10 & 7 & 9 & 6 & 5 & 11 & 12 & 13 & 1 \end{pmatrix}$

$\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 89 & 10 & 11 & 12 & 13 \\ 10 & 11 & 12 & 13 & 5 & 6 & 7 & 9 & 1 & 2 & 3 & 4 \end{pmatrix}$

$\delta\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 89 & 10 & 11 & 12 & 13 \\ 3 & 4 & 10 & 11 & 6 & 5 & 9 & 7 & 12 & 13 & 1 & 2 \end{pmatrix}$

$(\delta\gamma)^2 = \binom{1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 89 \quad 10 \quad 11 \quad 12 \quad 13}{10 \quad 11 \quad 12 \quad 13 \quad 5 \quad 6 \quad 7 \quad 9 \quad 1 \quad 2 \quad 3 \quad 4}$

$\Rightarrow \alpha\beta = (\delta\gamma)^2.$

Hence $\alpha\beta$ is a perfect product of two squares.

4 Conclusion

In conclusion, this work demonstrates the conditions under which the product of two idempotent elements in a finite full transformation semigroup is a perfect product of two squares. The key findings reveal that the kernel, image, and domain properties of these elements play a crucial role in determining when their product exhibits this perfect square structure. Additionally, our analysis of disjoint cycles provides a deeper understanding of the interaction between the square roots of transformation semigroup elements. The results presented here not only enrich the theoretical framework of transformation semigroups but also lay the groundwork for future research on related semigroup decompositions. The study of perfect products of squares offers potential applications in various areas of mathematics, including algebraic structures, combinatorics, and theoretical computer science, where semigroups and their decompositions are of central importance.

5 Recommendations

The results obtained in this work contribute significantly to semigroup theory, particularly in the study of transformation semigroups. The exploration of the perfect product of two squares within the finite full transformation semigroup $T(X)$ provides a deeper understanding of algebraic properties associated with these structures. However, further investigations can enhance the scope and impact of this study.

- **Incorporation of Rank Considerations:** One potential extension of this research is the analysis of the rank of transformations and its influence on whether a transformation can be expressed as a product of two squares. The rank of a transformation, which is the cardinality of its image, plays a crucial role in understanding transformation semigroups. Investigating how rank constraints affect the formation of square products can lead to a more comprehensive characterization of these elements.

- **Classification of Transformations Based on Rank:** A systematic classification of transformations in $T(X)$ based on rank and their decomposition into square products can provide a more structured approach to understanding semigroup composition. This classification may reveal deeper algebraic patterns and potential applications in computational semigroup theory.

- **Exploration of Generalized Square Products:** While this study focuses on perfect products of two squares, future research can examine the properties of multiple square compositions and their implications in transformation semigroups. Extending the analysis to higher-order power products could yield interesting algebraic results.

- **Application to Related Semigroups:** The techniques and results obtained here could be extended to other semigroups, such as the symmetric inverse semigroup or other transformation-related algebraic structures. Studying the existence and behavior of square products in these broader settings may provide a unified framework for semigroup decomposition.

By addressing these areas in future research, the study of transformation semigroups can be further enriched, providing a more complete understanding of their structural properties and expanding their theoretical and practical significance.

International Journal of Mathematical Sciences and Optimization: Theory and Applications
11(1), 2025, Pages 107 - 113
https://doi.org/10.5281/zenodo.15176079

6 Acknowledgment

The authors sincerely appreciate the support and contributions of the Mathematics Department, Faculty of Physical Sciences, Ahmadu Bello University, Zaria, for providing the academic environment that facilitated this research. We also extend our gratitude to colleagues and reviewers for their valuable feedback, which significantly improved the quality of this work. Special thanks go to the editorial board of the *International Journal of Mathematical Analysis and Optimization: Theory and Applications* for their insightful comments and recommendations.

7 Competing Interests

The authors declare that they have no competing interests.

References

[1] Snowden, M., and Howie, J. M. (1982). Square roots in finite full transformation semigroups. Glasgow Mathematical Journal, 23(2), 137–149. Cambridge University Press.

[2] Higgins, P. M. (1988). Digraphs and the semigroup of all functions on a finite set. Glasgow Mathematical Journal, 30(1), 41–57. Cambridge University Press.

[3] Higgins, P. M. (1986). A method for constructing square roots in finite full transformation semigroups. Canadian Mathematical Bulletin, 29(3), 344–351. Cambridge University Press.

[4] Annin, S., Cannon, T., Hernandez, C., and Torrel, L. (2012). On kth roots in the symmetric inverse monoid. Pi Mu Epsilon Journal, 321–331. JSTOR.

[5] Clifford, A. H., and Preston, G. B. (1961). The algebraic theory of semigroups, vol. 1. AMS Surveys, 7, 1967.

[6] Howie, J. M. (1995). Fundamentals of semigroup theory (Vol. 12). Oxford University Press.

[7] Howie, J. M. (1980). Products of idempotents in a finite full transformation semigroup. Proceedings of the Royal Society of Edinburgh, 86A, 243–254.

[8] Howie, J. M. (1966). The subsemigroup generated by the idempotents of a full transformation semigroup. Journal of the London Mathematical Society, 1(1), 707–716. Oxford University Press.

[9] Imam, A. T., Usman, L., Idris, A., & Ibrahim, S. (2024). Product of quasi-idempotents in finite semigroup of partial order-preserving transformations. International Journal of Mathematical Sciences and Optimization: Theory and Applications, 10(1), 53–59.

[10] Ibrahim, A., Nwaibeh, E. A., Francis, M. O., & Okigbo, C. J. (2024). On collapse of order-preserving and idempotent of order-preserving full contraction transformation semigroup. International Journal of Mathematical Sciences and Optimization: Theory and Applications, 10(2), 1–8.