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On Quasi-Nilpotents in Finite Partial Transformation Semigroups

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Abstract

Let X_n be the finite set $\{1, 2, 3 \cdots, n\}$ and $\mathcal{P}_n$ be the partial transformation on X_n. A transformation α in $\mathcal{P}_n$ is called quasi-nilpotent if when α is raised to some certain power it reduces to a constant map i.e α^m reduces to a constant map for $m \geq 1$. We characterize quasi-nilpotents in $\mathcal{P}_n$ and show that the semigroup $\mathcal{P}_n$ is quasi-nilpotent generated. Moreover if $K(n, r)$ is the subsemigroup of $\mathcal{P}_n$ consisting of all elements of height r or less, where height of an element α is defined as $|im\alpha|$, we obtained quasi-nilpotents rank of $K(n, r)$ that is the cardinality of a minimum quasi-nilpotents generating set for $\mathcal{P}_n$ as the stirling number of the second kind $S(n + 1, r + 1)$ which is the same as its idempotents rank.

Keywords: Nilpotents, Quasi-Nilpotent, Pseudo-Quasi-Nilpotent, Generating Sets, Rank.
MSC2010: 03G27, 16N40.

1 Introduction

Let $\mathcal{P}_n$ be the semigroup of partial transformation on a finite set $X_n = \{1, 2, 3 \cdots, n\}$. Evseev and Podran [2] considered the semigroup $\mathcal{P}_n$ consisting of all partial maps and showed that all elements other than permutations are expressible as products of idempotents. Garba [4] generalised the work of Evseev and Podran [2] to the ideals
$K(n, r) = \{\alpha \in \mathcal{P}_n : |im\alpha| \leq r\}$ for $2 \leq r \leq n - 1$ and studied product of idempotents as well as the idempotents rank in the partial transformation semigroup by using the result of Vagner [12] which says there is an isomorphism between $\mathcal{P}_n$ and a subsemigroup $\mathcal{P}_n^*$ of the full transformation semigroup $U_n = \{0, 1, 2 \cdots, n\}$ consisting of all maps α:$\{0, 1, 2 \cdots, n\} \mapsto \{0, 1, 2 \cdots, n\}$ for which $0\alpha = 0$.

Let S be a semigroup and let $\emptyset \neq A \subseteq S$, the smallest subsemigroup of S containing A is called the subsemigroup generated by A and is denoted by $\langle A \rangle$. Clearly $\langle A \rangle$ is the set of all finite products

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of elements of A. If there exists a non-empty subset A of S such that $S = \langle A \rangle$, then A is called a generating set of S. Also the *rank* of a finitely generated semigroup S is defined as

$$rank(S) = min\{|A| : A \subseteq S \text{ and } \langle A \rangle = S\}.$$

An element $\alpha \in \mathcal{P}_n$ $(\mathcal{T}_n)$ is called a quasi-nilpotent if when raised to some certain power reduces to a constant map and if S is generated by the set of only Quasi-nilpotent (QN) elements then the quasi-nilpotent rank is defined by

$$qnilrank(S) = min\{|A| : A \subseteq QN \text{ and } \langle A \rangle = S\}$$

The concept of quasi-idempotents elements was studied by Umar [11] and showed that $I_n^- = \{\alpha \in I_n : (\forall x \in dom\alpha,\ x\alpha \le x\}$ the set of all decreasing maps in symmetric inverse semigroup is quasi-idempotents generated and its rank is equal to $\frac{n(n+1)}{2}$. Madu and Garba [10] proved that each element in the semigroup $\mathcal{IO}_n = \{\alpha \in \mathcal{IO}_n : (\forall x, y \in dom\alpha,\ x \le y \implies x\alpha \le y\alpha\}$ the set of all order preservig maps in symmetric inverse semigroup is expressible as a product of quasi-idempotents of defect one in $\mathcal{IO}_n$ and that the quasi-idempotent rank and depth of $\mathcal{IO}_n$ are $2(n-1)$ and $(n-1)$ respectively. Garba *et al* [5] proved that $Sing_n$ is quasi-idempotent generated and that the quasi-idempotent rank of $Sing_n$ is $\frac{n(n-1)}{2}$. Imam *et al* [9] studied quasi-idempotet in finite partial order-preservig transformation semigroup $\mathcal{PO}_n$ and showed that the semigroup is quasi-idempotent generated and they also obtained an upper bound for quasi-idempotent rank of $\mathcal{PO}_n$ to be $\lceil \frac{5n-4}{2} \rceil$. Ali *et al* [1] investigated some rank properties on the semigroup of order-preserving order-decreasing partial contraction mappings on a finite chain $\mathcal{ODCP}_n$ and found the rank of $\mathcal{ODCP}_n$ to be $2n$.
Garba and Madu [3] studied Quasi-nilpotent elements in finite full transformation semigroup and showed that $sing_n$ is quasi-nilpotent generated and that the quasi-nilpotent rank is equal to $S(n,r)$ stirling number of the second kind which is the same as its idempotent rank.

In this article, we prove a result corresponding to that of Garba and Madu [3] for the partial transformations and that the quasi-nilpotents rank defined as the cardinality of a minimal generating set of quasi-nilpotents of $K(n,r)$ is $S(n+1, r+1)$, stirling number of the second kind.

2 Preliminaries

Let X be a finite set $\{1, 2, 3 \cdots, n\}$ and let U_n be the semigroup of all full transformation on X_0 where $X_0 = X \cup \{0\}$, by Vagner [12] For each α in P_n, we have the transformation α^* of X_0 by

$$x\alpha^* = \begin{cases} x\alpha \text{ if } x \in dom\alpha \\ 0 \quad \text{otherwise.} \end{cases}$$

Then α^* belongs to the subsemigroup P_n^* of U_n consisting of all those transformation of X_0 leaving 0 fixed. Conversely, if $\beta \in P_n^*$ then its restriction to X, $\beta X = \beta \cap (X \times X)$ is a partial transformation of X. The domain of $\beta \setminus X_n$ is the set of all x in X for which $x\beta \neq 0$. Then the mapping $\alpha \to \alpha^*$ and $\to \beta \setminus X$ are mutually inverse isomorphism of P_n onto P_n^* and vice-versa.

Lemma 2.1 (Garba and Madu (2003))**.** *: An element $\alpha \in Sing_n$ is quasi-nilpotent if and only if fix(α)= 1 and $A\alpha \neq A$ except fix(α) where A is a non-empty subset of X_n*

Lemma 2.2. *Every nilpotent element of height r in P_n correspond to a quasi-nilpotent in P_n^* under the Vagner isomorphism.*

Proof. Let $\alpha = \begin{pmatrix} A_1 & A_2 \cdots A_r \\ b_1 & b_2 \cdots b_r \end{pmatrix}$ be a nilpotent element in $\mathcal{P}_n$ of height r where $b_i \notin A_i$ and $A_i \in X_n$ for all i, then by definition of nilpotent element $fix(\alpha) = 0$ and $A\alpha \neq A$ where A is a

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non-empty subset of X_n, then clearly the image of $\alpha \in \mathcal{P}_n$ under the Vagner mapping representation is $\alpha^* \in \mathcal{P}_n^*$ where

$$\alpha^* = \begin{pmatrix} A_0 & A_1 & A_2 \cdots A_r \\ 0 & b_1 & b_2 \cdots b_r \end{pmatrix}$$

now α^* is a quasi-nilpotent since $fix(\alpha) = 1$ and $A\alpha \neq A$ except $fix(\alpha)$ and so the result follows from lemma 2.1 above. □

For example $\alpha = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$ is a nilpotent in P_3 and it correspond to $\alpha^* = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 0 & 1 \end{pmatrix}$ in P_3^*, because 2 is missing in the domain of α and by vagner [12], any element missing in the domain should be mapped to 0 and also 0 is mapped to itself.

From the work of Garba [4] (Lemma 2.4): P_n^* is a regular subsemigroup of U_n. and By Howie [6], $\mathcal{L}(P_n^*) = \mathcal{L}(U_n) \cap (P_n^* \times P_n^*)$, $\mathcal{R}(P_n^*) = \mathcal{R}(U_n) \cap (P_n^* \times P_n^*)$ and $\mathcal{J}(P_n^*) = \mathcal{J}(U_n) \cap (P_n^* \times P_n^*)$.

A typical $\mathcal{J}$-class of U_n (consisting of elements of height r $(1 \leq r \leq n+1)$) has $S(n+1, r)$ $\mathcal{R}$-classes, $\binom{n+1}{r}$ $\mathcal{L}$-classes, and each $\mathcal{L}$-class corresponds to a subset of U_n of cardinality r. Not every $\mathcal{L}$-class intersect P_n^*. In fact an $\mathcal{L}$-class intersect P_n^* if and only if its corresponding subset contains 0. So there are $\binom{n}{r-1}$ $\mathcal{L}$-classes containing 0, and $\binom{n}{r}$-classes not containing 0. (Observe as a check that $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$.
By contrast, every $\mathcal{R}$-class intersect P_n^*, since for a given equivalence ρ on U_n with r classes, we can choose an α such that $(0\rho)\alpha = 0$ and all other classes map in an arbitrary way.

Definition 2.3. *An element $\alpha^* \in \mathcal{P}_n^*$ is called a Pseudo-Quasi-Nilpotent (PQN) if $\alpha \in \mathcal{P}_n$ is a quasi-nilpotent. Where $\alpha^* \in P_n^*$ is the image of $\alpha \in \mathcal{P}_n$ under Vagner mapping representation.*

Lemma 2.4. *Every quasi-nilpotent element in $\mathcal{P}_n$ corresponds to a pseudo-quasi-nilpotent in $\mathcal{P}_n^*$ under Vagner mapping representation.*

Proof. Suppose $\alpha \in \mathcal{P}_n$ is a quasi-nilpotent element, then by definition $fix(\alpha) = 1$ and no non-empty subset A of X_n such that $A\alpha = A$ except $fix(\alpha)$, then the image of $\alpha \in \mathcal{P}_n$ under Vagner mapping representation is $\alpha^* \in \mathcal{P}_n^*$ and so clearly $fix(\alpha^*) = 2$ and thus $\alpha^* \in \mathcal{P}_n^*$ becomes a pseudo-quasi-nilpotent element. □

Lemma 2.5 (Garba(1990))**.** *: Every element α in P_n^* of height r is expressible as a product of idempotents of height r. that is $\alpha = \varepsilon_1, \varepsilon_2, ..., \varepsilon_m$, thus the subsemigroup P_n^* is idempotent generated.*

The next result gives a complete characterization of Pseudo-quasi-nilpotent elements in P_n^*.

Theorem 2.6. *An element $\alpha^* \in \mathcal{P}_n^*$ is pseudo-quasi-nilpotent if and only if:*

i $fix(\alpha^*) = 2$ that is $fix(\alpha^*) = \{0, i\}$ for some i in X_n

ii No non-empty subset A of X_n such that $A\alpha^* = A$ except $fix(\alpha^*)$.

Proof. Let $\alpha^* \in \mathcal{P}_n^*$, if $fix(\alpha^*) = 1$, then $fix(\alpha) = 0$ under the Vagner mapping representation and by lemma 2.3, α cannot be reduced to a constant mapping and so α is not a quasi-nilpotent element and as such α^* cannot be a pseudo-quasi-nilpotent. If $fix(\alpha^*) \geq 3$, then clearly the image of α under Vagner mapping representation will have $fix(\alpha) \geq 2$ and so by lemma 2.3, α cannot be a quasi-nilpotent element.

Conversely, suppose that $fix(\alpha^*) = 2$ that is $fix(\alpha^*) = \{0, i\}$ for some i in X_n and that No non-empty subset A of X_n such that $A\alpha^* = A$ except $fix(\alpha^*)$, then by lemma 2.3, α can be reduced to a constant mapping and so α becomes a quasi-nilpotent element. □

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3 Products of pseudo-quasi-nilpotents

We begin this section with a result that shows every idempotent in partial transformation Semigroup can be written in terms of two pseudo-quasi-nilpotents.

Theorem 3.1. *Every Idempotent $\beta^* \in \mathcal{P}_n^*$ is expressible as a product of two pseudo-quasi-nilpotent.*

Proof. Let $\beta^* \in \mathcal{P}_n^*$ be an idempotent of height r, that is $|im\beta^*| = r$, then we write β^* as $\beta^* = \begin{pmatrix} A_0 & A_1 & A_2 \cdots A_r \\ 0 & a_1 & a_2 \cdots a_r \end{pmatrix}$ where $a_i \in A_i$ for $0 \leq i \leq r$, Without loss of generality, we may assume two cases, when $|A_i| = 1$ for all i ≥ 1 and $|A_i| \geq 2$ for some i ≥ 1:
Case 1: If $|A_i| \geq 2$ for some $i \geq 1$ that is $A_1 = \{a_1, a_1', \cdots\}$, then

$$\beta^* = \begin{pmatrix} A_0 & A_1 & A_2 \cdots A_{r-1} & A_r \\ 0 & a_1 & a_3 \cdots a_r & a_1' \end{pmatrix} \begin{pmatrix} 0 & X_n \setminus Z & a_3 & a_4 & \cdots & a_r & a_1' \\ 0 & a_1 & a_2 & a_3 & \cdots & a_{r-1} & a_r \end{pmatrix}$$

Where $Z = \{a_3, a_4 \cdots, a_r, a_1',\}$, then clearly the two mappings on the right hand side are pseudo-quasi-nilpotents.
Case 2: If $|A_i| = 1$ for all i, then clearly the block A_0 must contain more than one elements , that is $|A_0| > 1$, since the mapping is not a permutation and so we can pick our a_1', from the block A_0 to be any non-zero element and the result follows from the first case. □

We now give two examples to explain the two cases considered.

Example 3.2. *for Case 1: Let $\beta^* = \begin{pmatrix} \{034\} & \{12\} & \{57\} & \{86\} & \{9\} \\ 0 & 2 & 7 & 6 & 9 \end{pmatrix}$ be an idempotent in $\mathcal{P}_9^*$. Since $|A_i| \geq 2$ for $i \geq 1$, then we have $Z = \{6, 9, 1\}$ and $X_n \setminus Z = \{2, 3, 4, 5, 7, 8\}$. then*

$$\beta^* = \begin{pmatrix} \{034\} & \{12\} & \{57\} & \{86\} & \{9\} \\ 0 & 2 & 6 & 9 & 1 \end{pmatrix} \begin{pmatrix} \{0\} & \{234578\} & \{6\} & \{9\} & \{1\} \\ 0 & 2 & 7 & 6 & 9 \end{pmatrix}$$

clearly the two mappings on the right are pseudo-quasi-nilpotent elements.
For Case 2: Let $\beta^* = \begin{pmatrix} \{0123\} & \{4\} & \{5\} & \{6\} & \{7\} & \{8\} \\ 0 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}$ be an idempotent in $\mathcal{P}_8^*$. Then $Z = \{6, 7, 8, 2\}$ where $2 \in A_0$ is the choosen non-zero element from the block A_0 and $X_n \setminus Z = \{1, 3, 4, 5\}$, now $\alpha^* = \begin{pmatrix} \{0123\} & \{4\} & \{5\} & \{6\} & \{7\} & \{8\} \\ 0 & 4 & 6 & 7 & 8 & 2 \end{pmatrix} \begin{pmatrix} \{0\} & \{1345\} & \{6\} & \{7\} & \{8\} & 2 \\ 0 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}$

Clearly the two mappings on the right hand side are pseudo-quasi-nilpotent elements.

Remark: Products of two Pseudo-quasi-nilpotent elements need not be a pseudo-quasi-nilpotent indicating that the pseudo-quasi-nilpotent do not generate themselves.

Example 3.3. *Let $\alpha^* = \begin{pmatrix} \{01\} & \{2\} & \{3\} \\ 0 & 1 & 3 \end{pmatrix}$ and $\beta^* = \begin{pmatrix} \{01\} & \{2\} & \{3\} \\ 0 & 2 & 1 \end{pmatrix}$ be two elements in $\mathcal{P}_3^*$ then both α^* and β^* are pseudo-quasi-nilpotent, but*

$\alpha^*\beta^* = \begin{pmatrix} \{012\} & \{3\} \\ 0 & 1 \end{pmatrix}$ *is not a pseudo-quasi-nilpotent.*

Theorem 3.4. *The Subsemigroup $\mathcal{P}_n^*$ of U_n consisting of all those transformations of X_0 leaving 0 fixed is pseudo-quasi-nilpotent generated.*

Proof. By Lemma 2.4, Every element $\alpha^* \in \mathcal{P}_n^*$ of height r is expressible as a product of idempotents of height r that is $\alpha^* = \epsilon_1^*, \epsilon_2^*, \cdots, \epsilon_m^*$ Thus the Subsemigroup $\mathcal{P}_n^*$ is idempotent generated and since every idempotent in the Subsemigroup $\mathcal{P}_n^*$ is expressible as product of two pseudo-quasi-nilpotent elements, then the result follows from theorem 3.1 □

Corollary 3.5. *The semigroup of partial transformation $\mathcal{P}_n$ is Quasi-nilpotent generated.*

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4 Pseudo-Quasi-Nilpotents Rank

We know that from the work of Garba and Madu [3], that for $1 \leq r \leq n$, let $K(n,r) = \{\alpha \in T_n : |im\alpha| \leq r\}$ then $K(n,n) = T_n$ and $K(n,n-1) = sing_n$. Each $K(n,r)$ is a two sided ideal of T_n and $K(n,r)/K(n,r-1)$ is a principal factor of T_n, which we can denote by P_r.

The quasi-nilpotent rank of the Semigroup $K(n,r) = S(n,r)$ i.e the sterling number of the second kind for all $n \geq 3$ and for all r such that $2 \leq r \leq n-1$. Now we wish to investigate the Pseudo-quasi-nilpotent rank in the partial transformation case $K^*(n,r)$ by applying the Vagner mapping representation.

Theorem 4.1. *The pseudo-quasi-nilpotent rank of the Semigroup $K^*(n+1,r)$ is $S(n+1,r)$ for $r \geq 2$*

Proof. Firstly, we would like to have a result like [lemma 5 of Garba and Madu [3]] but referring to elements of P_n^* (i.e elements preserving zero (0)) which says "let $A_1, \cdots, A_m$ (where $m = \binom{n}{r-1}$ and $r \geq 3$) be a list of the $0-subsets$ of X_0 with cardinality r, that is subsets of X_0 containing 0. Suppose there exist distinct equivalences $\pi_1, \pi_2, \cdots, \pi_m$ of weight r on X_0 satisfying the two conditions below

i A_{i-1}, A_i are both transversal of $\pi_i (i = 2, 3, \cdots, m)$ and A_m, A_1 are both transversal of π_1

ii A_{i+1} is not a transversal of π_i and A_1 is not a transversal of π_m

Then each non-group $\mathcal{H}$-class (π_i, A_{i+1}) and (π_m, A_1) contains a pseudo-quasi-nilpotent elements $\alpha_{i,i+1}$ and $\alpha_{m,1}$ respectively and there exist pseudo-quasi-nilpotent $q_{m+1}, \cdots, q_p$ (where $p = S(n+1,r)$) in P_n^* such that
$\{\alpha_{1,2}, \alpha_{2,3}, \cdots, \alpha_{m-1,m}, \alpha_{m,1}, q_{m+1}, \cdots q_p\}$ is a set of generators for $K^*(n+1,r)$."

By Howie and Mcfadden [8] each $\mathcal{H}$-class (π_i, A_i) contains idempotent ϵ_i and there exist idempotents $(\epsilon_{m+1}, \epsilon_{m+2}, \cdots, \epsilon_p)$ where $p = S(n,r)$ such that $\{\epsilon_1, \epsilon_2, \cdots, \epsilon_p\}$ is a set of generators for $K(n,r)$. The idempotents $\epsilon_{m+1}, \cdots, \epsilon_p$ are chosen so that $\{\epsilon_1, \cdots, \epsilon_p\}$ cover the $\mathcal{R}$-classes in P_r. Our objective is to show that $\{\epsilon_1, \cdots, \epsilon_p\} \subseteq \langle \alpha_{1,2}, \cdots, \alpha_{m-1,m}, \alpha_{m,1}, q_{m+1}, \cdots, q_p \rangle$. We choose the pseudo-quasi-nilpotent $q_{m+1}, \cdots, q_p$ so that $\alpha_{1,2}, \cdots, \alpha_{m-1,m}, \alpha_{m,1}, q_{m+1}, \cdots, q_p$ cover all the $\mathcal{R}$-classes in P_r. □

To prove the listing of images and kernel equivalences postulated in the statement above can actually be carried out. Let $n \geq 3$ and $2 \geq r \geq n-1$ and consider the following proposition which is the $\mathcal{P}_n^*$ version of the equivalent lemma 5 in Garba and Madu [3] which says "There is a way of listing the 0-subsets of X_0 of cardinality r as $A_1, \cdots, A_m$ (with $m = \binom{n}{r-1}$),

$A_1 = \{0, 1, \cdots, r-1\}, A_m = \{0, n-r+2, \cdots, n\}$) so that there exist equivalences $\pi_1, \cdots, \pi_m$ of weight r with the property that

i A_{i-1}, A_i are both transversal of $\pi_i (i = 2, 3, \cdots, m)$ and A_m, A_1 are both transversal of π_1

ii A_{i+1} is not a transversal of π_i and A_1 is not a transversal of π_m

Then all we have to do here is to consider $\{0\}$ as a singleton class for each $\pi_i (i = 1, 2, \cdots, m)$ and the rest of the π_i-classes as in Garba and Madu [3]. To exemplify the process, We Let $n = 4$ and consider the set $\{1, 2, 3, 4\}$. The list given in Garba and Madu [3] of all the subsets of cardinality 2 with the equivalences is given below:

$$A'_1 = \{1, 2\} \quad \pi'_1 = 13 \setminus 24$$
$$A'_2 = \{1, 3\} \quad \pi'_2 = 14 \setminus 23$$
$$A'_3 = \{1, 4\} \quad \pi'_3 = 12 \setminus 34$$
$$A'_4 = \{2, 3\} \quad \pi'_4 = 124 \setminus 3$$

$$A'_5 = \{2, 4\} \quad \pi'_5 = 134 \setminus 2$$
$$A'_6 = \{3, 4\} \quad \pi'_6 = 123 \setminus 4$$

Now for the set $\{0, 1, 2, 3, 4\}$ we have the 0-subsets of cardinality 3 and the equivalences as given below

$$A_1 = \{0, 1, 2\} \quad \pi_1 = 0 \setminus 13 \setminus 24$$
$$A_2 = \{0, 1, 3\} \quad \pi_2 = 0 \setminus 14 \setminus 23$$
$$A_3 = \{0, 1, 4\} \quad \pi_3 = 0 \setminus 12 \setminus 34$$
$$A_4 = \{0, 2, 3\} \quad \pi_4 = 0 \setminus 124 \setminus 3$$
$$A_5 = \{0, 2, 4\} \quad \pi_5 = 0 \setminus 134 \setminus 2$$
$$A_6 = \{0, 3, 4\} \quad \pi_6 = 0 \setminus 123 \setminus 4$$

Theorem 4.2. *The Pseudo-quasi-nilpotent rank of $K(n, r)$ is equal to its idempotent rank which is the stirling number of the second kind $S(n + 1, r + 1)$ for $2 \leq r \leq n - 1$ where $K(n, r) = \{\alpha \in P_n : |im\alpha| \leq r\}$.*

Proof. By the isomorphism between $\mathcal{P}_n$ and $\mathcal{P}_n^*$ elements of height r in $\mathcal{P}_n$ correspond to element of height $r + 1$ in $\mathcal{P}_n^*$. It therefore follows that the image of $K(n, r)$ under the isomorphism is $K^*(n + 1, r + 1)$ and the result follows from theorem 4.1 above. □

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