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On the Fixed Points of Springer Varieties in Type A

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Abstract

Springer varieties are sub-varieties of the full (complete) flag variety $\mathcal{F}\ell_n(\mathbb{C})$, which can be thought of as the fiber over X of the Springer resolution of singularities of cone of nilpotent endomorphism $X : V \longrightarrow V$, where X is a nilpotent endomorphism in its Jordan canonical form of type λ and V is a $n-$dimensional vector space over $\mathbb{C}$ ($V = \mathbb{C}^n$ for convenience). The geometry of Springer varieties is reviewed in this article, along with their S^k- fixed points. We accomplish this by briefly reviewing nilpotent orbits in type A within the framework of integer partition.

Keywords: Flag Variety, Springer Varieties, Bruhat Order, Coxeter Group, General Linear Group.
MSC2010: 14M15, 14N15, 05E05.

1 Introduction and Preliminaries

Fixed points play an elegant role in various areas of Mathematics, including, Dynamical systems, Optimization [1], and Topology [2]. In this article, we gives a review of the S^k-fixed points of the Springer varieties, which are members of the family of Hessenberg varieties, which are also the family of closed subvarieties of the full flag variety. Hessenberg varieties in [3] was introduced by De Mari and Shayman in the late $1980's$ in order to efficiently compute the eigenvalues and eigen space of a linear operator in question related to numerical analysis. Ever since their introduction, they have been objects of active research in Combinatorics, Geometry, Representation theory and Topology.

For instance, Tymoczko [4] showed that Hessenberg Varieties are not always pure dimensional. Jordan and Helmke [5] proved that the QR-algorithm restricted on the subset of Hessenberg varieties is generically controllable. In the special case of Peterson varieties Pet_n (type A), Adeyemo in [6] studied the combinatorial aspect of the Peterson varieties where he established its Galois connection with Boolean poset and showed that its Hasse diagram admits EL- labeling and also show that it is lexicographically shellable and hence Cohen macauley. Yukiko Fukukuwa, Harada and Masuda [7] gave an efficient presentation of S^1-equivariant cohomology ring of Peterson variety in type A as a quotient of a polynomial ring. Insko [8] used the tight correlation between the geometry of the Peterson variety and the combinatorics of the symmetric group to prove that the homology of the

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Peterson variety injects into the homology of the flag variety. Insko and Yong [9] described the singular locus of the Peterson variety and gave explicit equivalent K-theory localization formula. Harada and Tymoczko [10] identified a computationally convenient basis of $H^*_{S^1}(Pet_n)$ which they named basis of Peterson shcubert classes, they also derived a manifestly positive, integral Chevalley-Monk formula for the product of a cohomology-degree-2 Peterson Schubert class with an arbitrary Peterson Schubert class. Darius and Megumi [11] Used the Monk formula derived in the work of Megumi and Tymoczko above to prove a Gaimbelli formula for the Peterson Schubert classes in the S^1-equivariant cohomology ring of a type A Peterson variety. Hariku , etal [12] gave a systematic method for producing an explicit presentation by generators and relations of the equivariant and ordinary cohomology rings (with coefficients in $\mathbb{Q}$) of any regular nilpotent Hessenberg variety in type A. Megumi and Tymoczko Introduced a combinatorial game which they called Poset Pinball and construct computationally convenient module bases for the S^1-equivariant cohomology of all Peterson varieties of classical Lie type and subregular Springer varieties of Lie type A. Darius and Megumi [11] developed the theory of poset pinball introduced by Megumi and Tymoczko for the study of the equivariant cohomology rings of GKM-compatible subspaces of GKM space.
In our direction (Springer varieties), Precup and Tymoczko [13] related Springer and Schubert varieties combinatorially and proved that the Betti numbers of the Springer varieties associated to a partition with at most three rows or two columns are equal to the Betti numbers of a specific union of Schubert varieties. Fresse [14], via an algorithm, determined the Betti numbers of the Springer varieties through cell-decomposition of the springer varieties. The algorithms used in [15] and [14] to compute the Betti number of Springer varieties were compared in [16]. Fresse [17] established a necessary and sufficient condition of singularity for the components of the Springer varieties. Horiguchi [18] gave an explicit presentation of the S^1-equivariant cohomology ring of two row $(n-k,k)$ Springer varieties (in type A) as a quotient of polynomial ring in an ideal I. Hiraku and Horiguchi [18] gave a presentation of the T^l-equivariant cohomology ring of the Springer varieties through an explicit construction of an action of the n^{th} symmetric group on the T^l-equivariant cohomology group. Tymoczko [19] described a way that Springer and Schubert varieties are related. Springer [20] noticed that the cohomology of Spr_λ admits the action of the group of permutations S_n. It was shown in [21] that Springer varieties has a partition into finite number of locally closed affine spaces, such that this partition is determined by the Young diagram associated to a nilpotent operator X. Fung in [22], described the irreducible components of Springer varieties for nilpotent operator X of Jordan type $\lambda = (n-k, 1^k)$ and $\mu = (2,2,2,\cdots,l)$, $1 \leq l \leq 2$ as an iterated bundles of the full flag varieties and Grassmannians and related the topology of pairwise intersection of these components with the inner product of the kazhdan-Lustig basis. A topological construction of Springer varieties corresponding to nilpotent operator with Jordan type $\lambda = (k,k)$ was given in [23]. Row strict tableaux were used in [14] to determined the Betti numbers of the Springer varieties through their cell-decomposition. Fresse in [17]considered Springer varieties, with the associated Young daiagram having exactly two columns and established a necessary and sufficient condition of singularity for the components of Springer varieties. Russel [24] gave an extension of this work by using the combinatorial and diagrammatic properties of Khovanov's construction to provide a useful homology basis and construct the Springer representation with this basis. It was shown in [25] that the unimodality of the distribution of Betti numbers of Springer varieties is limited to the nilpotent operators $X \in \mathfrak{gl}_n(\mathbb{C})$ of the following Jordan forms $\lambda = (\lambda_1, \lambda_2)$, $(\lambda_1, \lambda_2, 1)$, $(\lambda_1, 1^{\lambda_1})$ and $(\lambda_1^2, 1^{\lambda_1})$.
Horiguchi in [18] gave an explicit presentation of the S^1-equivariant cohomology ring of the two row $(n-k,k)$ Springer varieties (in type A) as a quotient of polynomial ring. Fresse, etal [26] studied the structure of the smooth irreducible components of Springer varieties and show that each smooth component has a structure of iterated bundles of Grassmannian. Hiraku, etal [12] gave a presentation of the $T-$equivariant cohomology ring of the Springer varieties through an explicit construction of an action of the n^{th} symmetric group on the $T-$equivariant cohomology group. [19] described a way the Springer and Schubert varieties are related. In [27], combinatorial connections were made between the Springer and Schubert varieties, and it was show that the Betti numbers of the Springer varieties associated with a partition with not more than three rows or two columns

are equivalent to the Betti numbers of a particular union of Schubert varieties.Invariants were attached to each standard Young tableaux in [28] to study the geometry of Springer varieties in type A.

The break down of this work is as follows: Section two contains some basic definitions and existing theorems of some tools, such as partition of integers, young diagram and tableaux, group of permutations as coxeter groups, which shall be needed often. In section three, we introduce the family of Hessenberg varieties and hence our main object of study which is Springer varieties. Since we work in type A, therefore, we quickly talk about the Lie algebra of the general linear group $GL_n(\mathbb{C})$ in section four. Section five contains some fact about the Grassmannians and flag variety. Classification of the nilpotent orbits in type A is discussed in section six. Resolution of singularities of the closure of the nilpotent orbit is considered in section seven. Finally, we talk about the fixed points of the Springer varieties section eight.

2 Basic Definitions and Existing Theorems

A partition λ of non negative integer n written as $\lambda \vdash n$ is a sequence of integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ and $\sum_{i=1}^{k} \lambda_i = n$, where each λ_i is called the parts of λ. The number of parts is the length of λ denote by $\ell(\lambda)$; and the sum of parts is the weight of λ denoted by $|\lambda| = \lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_k$.

Example 2.1. *Let $n = 5$, and $\lambda = (2, 2, 1)$ is a partition of 5, $\ell(\lambda) = 3$ and $|\lambda| = 5$*

We denote the set of all partitions of n by $\mathcal{P}_n$, and the set of partitions by $\mathcal{P}$. Sometimes we use a notation which indicates the number of times each integer occurs as a part:

$$\lambda = (a_1^{m_1}, a_2^{m_2}, \cdots a_k^{m_k}).$$

Note that the number of times a_i appears is m_i, where $1 \leq i \leq k$ and we referrer to m_i as the multiplicity of a_i in λ. For instance,

$$\lambda = (2, 2) = (2^2)$$

$$\lambda = (211) = (21^2).$$

Thus

$$\mathcal{P}_5 = \{(5), (4, 1), (3, 2), (3, 1^2), (2^2, 1), (2, 1^3)(1^5)\}.$$

Below is a table of $n \leq 14$ and its corresponding $|\mathcal{P}_n|$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\lvert\mathcal{P}_n\rvert$	1	2	3	4	7	11	15	22	30	42	56	77	101	135

.

We take $\mathcal{P}_n = 0$ for all $n < 0$ and $\mathcal{P}_0 = 1$. Integer partitions were first studied by Euler. For many years one of the most intriguing and difficult questions about them was determining the asymptotic properties of $\mathcal{P}_n$ as n gets large. Readers are encouraged to consult [29] for details on theory of partitions.

Remark 2.2. *The number of partitions in $\mathcal{P}_n$ increases quite rapidly with n. For example,$\mathcal{P}_{10} = 42$, $\mathcal{P}_{20} = 627$, $\mathcal{P}_{50} = 204226$, $\mathcal{P}_{100} = 190569292$, and $\mathcal{P}_{200} = 3972999029388$.*

Let $\mathcal{L}_n$ denote the reverse lexicographical ordering on the set $\mathcal{P}_n$: that is to say, $\mathcal{L}_n$ is the set of $\mathcal{P}_n \times \mathcal{P}_n$ consisting all (λ, μ) such that either $\lambda = \mu$ or the first non-vanishing difference $\lambda_i - \mu_i$ is positive [29]. $\mathcal{L}_n$ is a total ordering.

Example 2.3. *when $n = 5$, $\mathcal{L}_5$ arranges $\mathcal{P}_5$ in the sequence.*

$$(5), (4, 1), (3, 2), (3, 1^2), (2^2, 1), (2, 1^3), (1^5).$$

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We define another ordering $\mathcal{L}'_n$ on $\mathcal{P}_n$ as follows: The ordering $\mathcal{L}'_n$ on $\mathcal{P}_n$ is the set of all (λ, μ) such that either $\lambda = \mu$ or the first non-vanishing difference $\lambda^*_i - \mu^*_i$ is negative, where $\lambda^*_i = \lambda_{n-i+1}$ [29].

Theorem 2.4. *[30] Let $\lambda, \mu \in \mathcal{P}_n$. Then, $(\lambda, \mu) \in \mathcal{L}'_n \iff (\mu', \lambda') \in \mathcal{L}_n$.*

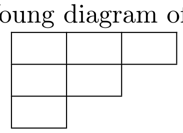
The dominance partial order on partitions of some fixed none-negative integer n written $\trianglerighteq$ is defined as follows.
If both λ and μ are partitions of n then $\lambda \trianglerighteq \mu$ if for all
$m \geqslant 1$, $\sum_{i=1}^{m} \lambda_i \geqslant \sum_{i=1}^{m} \mu_i$. holds. For example if $\lambda = (3, 2, 1)$, $\mu = (2, 2, 2)$ and $\nu = (3, 1, 1, 1)$ then $\lambda \trianglerighteq \nu$, $\lambda \trianglerighteq \mu$ and $\nu$and μ are incomparable.

Remark 2.5. *The orderings $\mathcal{L}_n, \mathcal{L}'_n$ are distinct for $n \geq 6$. For example, if $\lambda = (3, 1^3)$ and $\mu = (2^3)$ we have $(\lambda, \mu) \in \mathcal{L}_6$ and $(\mu, \lambda) \in \mathcal{L}'_6)$. Hence if $\lambda, \mu \in \mathcal{P}_n$, then $(\lambda, \mu) \in \mathcal{L}_n \iff (\mu, \lambda) \in \mathcal{L}'_6$.*

2.1 A Brief Survey of Some Theories of Young Tableaux

The Young diagram of an integer partition gives us an interesting and practical tool for visualising partitions of integers, and in most cases for establishing identities. It is a collection of cells (boxes) arranged in left justified rows such that the number of cells in each row corresponds to the size of a part and is weakly decreasing from top to bottom.
For instance, the Young diagram of $\lambda = (3, 2, 1)$ is shown in figure below.

Figure 1: Young diagram of $\lambda = (3, 2, 1)$



The size of a Young diagram is the total number of cells in the diagram. For instance, the size the above diagram is 5. Suppose λ is a partition of n. The length of row i in the corresponding Young diagram is λ_i. The column lengths λ'_i is equally a partition of n and it is called conjugate or dual partition. It is noted that λ'_1 is the number of nonzero parts of λ. If λ=$(4, 3, 1, 1)$ then λ' $= (4, 2, 2, 1)$.
Let $\lambda \vdash n$ and λ' its dual, then λ'_i = number of $\lambda_i \geq i$. In particular $\lambda'_1 = \ell(\lambda)$ and $\lambda_1 = l(\lambda')$. Obviously λ''= λ.
Filling the cells of a Young diagram with non-negative integers in accordance to some certain rules, results to a central objects of algebraic combinatorics and geometry, called, Young tableau(tableaux for plural).
A row strict tableau (rst) is a filling of a Young diagram, from $[n]$ whose entries strictly increase (from left to right) along the row,with no condition on the columns.

[28]

1	4	6	8	9	10
3	5	13	14	16	
2	12	15			
7	11				

Figure 2: row strict tableau

We denote by $(rst)^{\lambda}$ the collection of all row-strict tableaux of shape λ. A column strict tableau (cst) is a filling of a Young diagram, from $[n]$ whose entries strictly increase down the column,with no condition on the rows.

2	1	6	8	9	10
4	3	13	15	14	
5	11	16			
7	12				

Figure 3: column strict tableau

For $\lambda \vdash n$ we define a standard Young tableau (SYT) as a filling of a Young diagram of shape λ such that the integers from 1 to n appears exactly once and that its entries are increasing across each row and column. In other words, a standard tableau is a filling of a Young diagram that is both row and column strict. A filling of a Young diagram is called $(h, \lambda)-$filling if for any two

1	2	5	6	11	15
3	7	8	12	14	
4	9	16			
10	13				

Figure 4: standard tableau

adjacent entries of the form $\boxed{k \mid j}$, $k \leq h_j$. Where $h : [n] \longrightarrow [n]$ is a non-decreasing function called Hessenberg function is such that

i) $i \leq h_i \leq n, \forall i \in [n]$

ii) $h_i \leq h_{i+1}, \forall i \in [n-1]$

[15]

Example 2.6. *Let $n = 5$, $\lambda = (2, 2, 1)$ and $h = (1, 2, 3, 4, 5,)$ then*

3	4
2	5
1	

is an $(h, \lambda)-$filling since $k \leq h_j$ for each $\boxed{k \mid j}$

It is noted that if $h = (1, 2, 3, \cdots, n)$, $(h, \lambda)-$filling is the same as row strict.

The hook length h_{ij} of a given box (ij) in a frame of a young diagram of shape λ is the length of the right-angled path in the frame with that box as the upper left vertex (i labels rows j labels column). For example, the hook length of the asterisked box in

*	•	•	•	•
•				
•				
•				
•				

is 9. (i.e $h_{21} = 9$)

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Hook length formula (Frame, Robinson, and Thrall). If λ is a Young diagram with n boxes, then the number f^λ of standard tableaux with shape λ is $n!$ divided by product of the hook lengths of the boxes.i.e. if $\lambda \vdash n$, then

$$f^\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h_{i,j}}$$

. For example, to compute the number of standard Young tableaux of shape $\lambda = (2,2,1) \vdash 5$. The hooklengths are given in the array

$$\begin{array}{|c|c|}\hline 4 & 2 \\ \hline 3 & 1 \\ \hline 1 & \\ \cline{1-1}\end{array}$$

where $h_{i,j}$ is placed in (i,j). Thus

$$f^{221} = \frac{5!}{4\cdot 3\cdot 2\cdot 1^2} = 5.$$

Below is the list of all possible standard Young tableaux of shape $\lambda = (2,2,1)$:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \\ \cline{1-1}\end{array}$$

From the above, it is obvious to see that if $\lambda = n$, then $f^{(n)} = 1$ also, if $\lambda = 1^n$ then, $f^{(1^n)} = 1$. In a similar approach, let $(rst)_\lambda$ be the set of row strict tableaux of shape $\lambda = (\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_k)$, the cardinality of $(rst)_\lambda$ denoted by $|(rst)_\lambda|$ is given as

$$|(rst)_\lambda| = \frac{n!}{\prod_{i=1}^{k} \lambda_i!}. \tag{2.1}$$

Example 2.7. *Let $n = 5$ and $\lambda = (2,2,1)$ $|(rst)_\lambda| = \frac{5!}{2!\times\times 2!\times 1!} = 30$.*

2.2 Group of permutations as Coxeter Groups

Denote by S_n, the group of permutations of n letters, such that $S_n = \langle s_1, s_2, \cdots, s_{n-1}\rangle$ with the following relations:

- $s_i^2 = e\ \forall\ 1 \le i \le n-1$;
- $s_i s_j = s_j s_i$ if $|i-j| \ge 2$;
- $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},\ 1 \le i \le n-1$.

Where $s_i = (i, i+1)$ is a simple transposition and e is the identity element of S_n.
We shall denote $w \in S_n$ by the sequence $w_1 w_2 \cdots w_n$ which is the one-line notation of w. For instance, let $w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 5 & 4 \end{pmatrix} \in S_5$ is written as 13254. For each $w \in S_n$, the inversion of w is defined as inv(w)= $\{(i,j) : w_i > w_j \text{ and } 1 \le i < j \le n\}$ and the cardinality of inv(w) denoted by $l(w)$ and call it the length of w. For example, inv(13254)=$\{(2,3),(4,5))\}$ with $l(13254) = 2$.
In a similar manner, any $w \in S_n$ can be written as a product of generators:

$$w = s_1 s_2 \cdots s_k,\ \ s_i \in S_n.$$

If k is the minimal of all such expressions for w, then k is the length of w and we write $l(w) = k$. The word $s_1 s_2 \cdots s_k$ is called a reduced word, reduced decomposition or reduced expression for w

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which is not necessarily unique. It is established in various literatures, [6, 7, 33] (wehre details on this topic could be found) that the length of reduced word of w is equal to the cardinality of inv(w). One important fact about the group of permutation that is note worthy is the existence of a longest element. In oneline notation, it is of the form $n, n-1, \cdots, 2, 1$, and it has length $\frac{n(n-1)}{2}$ and we denote this by w_0

Proposition 2.8. *[31] Let $w \in S_n$. Then*

(1) $l(w) = 0$ *if and only if* $w = e$.

(ii) $l(w) = 1$ *if and only if* $w = s_i$ $(1 \leq i \leq n-1)$.

(iii) $l(w) = l(w^{-1})$.

(iv) Let $w_0 = (n, n-1, \cdots, 2, 1) \in S_n$. *Then* $l(w_0 w) = l(ww_0) = \frac{1}{2}n(n-1) - l(w)$.

2.2.1 Bruhat Order

Bruhat order is a partial order defined on S_n which is in agreement with length function.
Because of the Coxeter groups' usefulness in the partial order structure, which aids in their theory, algebraic combinatorists and geometers place a high importance on them, as it helps in obtaining the closure of a Schubert cell C_w for every $w \in S_n$.
We begin by defining a relation $\longrightarrow$ on S_n by saying that, for any $w, u \in S_n$ $u \longrightarrow w$ if there is a transposition (not necessarily simple) t such that $w = tu$ (or)$u^{-1}w = t$ and $l(u) < l(w)$. We write $u \leq w$ if there exists $u_i \in S_n$ such that
$u = u_0 \longrightarrow u_1 \longrightarrow \cdots \longrightarrow u_{k-1} \longrightarrow u_k = w$ and $l(w) = l(u) + k$.

Remark 2.9. *In a simple language, we say $u \leq w$ in the Bruhat order if for any reduced word decomposition of w there is at least a subword of w which equals u. For instance, if $w = s_1s_2s_1$ and $u = s_1s_2$ then $v \leq w$ in Bruhat order since s_1s_2 is a subword of $s_1s_2s_1$*

Proposition 2.10. *[31] For any $u, w \in S_n$ the following are equivalent:*

(i) $u \leq w$.

(ii) Every reduced expression of w has a subword that is reduced expression of u.

(iii) Some reduced expression of w has subword that is a reduced expression of u.

Group of permutations S_n is ubiquitous and quite useful in Mathematics. In algebra for instance, its representation theory has numerous applications in geometry, it parameterized the $CW-$decomposition of the full flag variety [10]. Also, it is useful in the combinatorics of integer partions and tableaux [10]. The description of the Schubert variety as the closure of its cell C_w (i.e $X_w = \overline{C}_w = \cup_{v \leq w} C_v$) leads to the inclusion relation of two Schubert varieties $X_w \leq X_u$ for $u, w, v \in S_n$ defines $w \leq u$, $v \leq w$ in the bruhat order which has many connections to combinatorics and representation theory. This leads to a strong interplay of Geometry, Combinatorics and Representation theory which makes Hessenberg varieties more attractive to researchers.

3 Hessenberg Varieties

Hessenberg varieties are parameterized by an operator $X : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ and a non decreasing function $h : [n] \longrightarrow [n]$, called Hessenberg function. where

$$[n] = \{1, 2, 3 \cdots, n\}, \tag{3.1}$$

such that

i) $h(i) \geq i$ for all $1 \leq i \leq n$,

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ii) $h(i+1) \geq h(i)$ for all $1 \leq i \leq n$.

The Hessenberg varieties associated to a nilpotent operator X and a Hessenberg function h denoted by $H(X,h)$, is defined as

$$H(X,h) = \{V_\bullet \in \mathcal{F}\ell_n(\mathbb{C}) : XV_i \subseteq V_{h(i)},\ \forall\, i\} \tag{3.2}$$

For a fixed nilpotent operator X, there are three cases for the choice of Hessenberg function h. Viz:

i) the first occurs when $h(i) = i$ for all i

ii) the second case occurs when $h(i) = i+1,\ 1 \leq i \leq n-1$

iii) and the case third occurs when $h(i) = n$ for all i.

In the first case, the obtained varieties are the Springer varieties which we denoted by Spr_λ, where $\lambda = (\lambda_1, \cdots, \lambda_k)$ corresponds to the sizes of Jordan blocks of X. In the second case. The second case yields Peterson variety, where in the third case, when $h(i) = n$, is the full flag variety.

Remark 3.1. *In other words, Springer varieties Spr_λ is the family of full flag in $\mathbb{C}^n$ fixed by nilpotent operator X with Jordan blocks determined by λ.*
In two opposites extremal cases, Spr_λ are irreducible:

i) If X has only one nontrivial block, i.e. $\lambda = (n)$, (in other words X is regular nilpotent) then Spr_λ consists of flags of the form $V_0 \subset V_1 \subset \cdots \subset V_n$ where $V_i = ker(X^i)$.

ii) At the other extreme, if $\lambda = (1^n)$, implying that $X = 0$, then Spr_λ coincides with the whole flag variety $\mathcal{F}\ell_n(\mathbb{C})$. .

In any other case, Spr_λ are reducible into irreducible components.

4 Lie Algebra of $GL_n(\mathbb{C})$

For the rest of this work, the underlying field shall be $\mathbb{C}$, and we take $GL_n(C) = G$, unless otherwise stated. Let $M_n(\mathbb{C})$ be the set of all $n \times n$ matrices whose entries are in $\mathbb{C}$. We denote the entries of any $A \in M_n(\mathbb{C})$ by $a_{i,j}$ and also write $A = [a_{i,j}]$. We denote the identity and zero matrix in $M_n(\mathbb{C})$ by I_n and 0_n respectively, and define $E_{i,j}$ with entry $e_{i,j} = 1$ and zero elsewhere. $E_{i,j}$ form a basis of $M_n(\mathbb{C})$. Hence, the dim$M_n(\mathbb{C})$=n^2, $M_n(\mathbb{C})$ is a ring with the usual addition and multiplication of $n \times n$ matrices. $M_n(\mathbb{C})$ is not commutative except $n = 1$.

Proposition 4.1. *[32] The determinant function $det : M_n(\mathbb{C}) \longrightarrow \mathbb{C}^*$ has the following properties:*

i) For $A, B \in M_n(\mathbb{C})$, $det(AB) = detAdetB$.

ii) $A \in M_n(\mathbb{C})$ is invertible if and only if $detA \neq 0$.

iii) $detI_n = 1$.

The collection of $A \in M_n(\mathbb{C})$ that satisfy item (ii) in the above proposition forms a group with respect to usual matrix multiplication, called the general linear group (also known as the set of units in the ring $M_n(\mathbb{C})$) denoted by $GL_n(\mathbb{C})$.
Bellow are some subgroups of $GL_n(\mathbb{C})$:

i) the spacial linear group $SL_n(\mathbb{C})$, defined as

$$SL_n(\mathbb{C}) = \{A \in GL_n(\mathbb{C}) : detA = 1\} \subset GL_n(\mathbb{C}). \tag{4.1}$$

ii) the unitary group denoted by $U_n = \{A \in GL_n(\mathbb{C}) : A^*A = I_n\}$. Where A^* is the complex conjugate of A.

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iii) The Borel subgroup B of $GL_n(\mathbb{C})$, defined as $B = \{A \in GL_n(\mathbb{C}) : A \text{ is Upper triangular}\}$

iv) the $n-$dimensional torus $T^n = \{g \in U_n : g = Diag(t_1, t_2, \cdots, t_n),\ |t_i| = 1,\ i = 1, 2, \cdots, n\}$.

Theorem 4.2. *[32] The sets $GL_n(\mathbb{C})$ and $SL_n(\mathbb{C})$ are groups under multiplication. Furthermore , $SL_n(\mathbb{C})) \leq GL_n(\mathbb{C})$.*

Remark 4.3. *$GL_n(\mathbb{C})$ is not just a group but a Lie group. The group structure of $GL_n(\mathbb{C})$ and $SL_n(\mathbb{C})$ leads to the name "general linear group", and "special linear group" respectively. There are other subgroup of $GL_n(\mathbb{C})$ which are not discussed here, readers are encouraged to consult [32] for details on matrix group.*

We like to recall at this juncture that the exponential of any $A \in M_n(\mathbb{C})$ denoted by e^A is defined as

$$e^A = \sum_{k=0} \frac{A^k}{k!}. \tag{4.2}$$

This series converges for each $A \in M_n(\mathbb{C})$.
Let A_n be a sequence of a complex matrices. We say that A_n converges to a matrix A if each entries of A_n converges to the corresponding entry of A.
A matrix Lie group is a subgroup H of $GL_n(\mathbb{C})$ with the property that if A_n is a sequence of a matrix in H and A_n converges to a matrix A, then either A belongs to H or A is not invertible. The above definition implies that H must be a closed subgroup of $GL_n(\mathbb{C})$.
A matrix Lie group G is connected if for each $E, F \in G$, there is a continuous path $A : [a, b] \longrightarrow G$ such that $A(t) \in G$ for each $t \in [a, b]$, $A(a) = E$ and $A(b) = F$.

Let G be a matrix Lie group. The Lie algebra of G denoted by $\mathtt{g}$ is the set of all matrices A such that $e^{\alpha A}$ belongs to G for each $\alpha \in \mathbb{R}$.

Example 4.4. *If $G = GL_n(\mathbb{C})$ then its corresponding Lie algebra is $\mathtt{gl}_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : e^{\alpha A} \in GL_n(\mathbb{C}),\ \alpha \in \mathbb{R}\}$.
Also if $G = SL_n(\mathbb{C})$ then*

$$\begin{aligned} \mathtt{sl}_n(\mathbb{C}) &= \{A \in M_n(\mathbb{C}) : e^{\alpha A} \in SL_n(\mathbb{C}),\ \alpha \in \mathbb{R}\} \\ &= \{A \in M_n(\mathbb{C}) : det(e^{\alpha A}) = 1,\ \alpha \in \mathbb{R}\} \\ &= \{A \in M_n(\mathbb{C}) : e^{\alpha tr(A)} = 1,\ \alpha \in \mathbb{R}\} \\ &= \{A \in M_n(\mathbb{C}) : tr(A) = 0\} \end{aligned}$$

Remark 4.5. *In addition to the above, $GL_n(\mathbb{C})$ is a connected Lie group of dimension n^2. The Lie algebra $\mathtt{gl_n}(\mathbb{C})$ corresponding to $GL_n(\mathbb{C})$ consists of all $n \times n$ complex matrices with the commutator $[.,.]$ called the Lie bracket. If $A \in M_n(\mathbb{C})$ we define the trace of A as $tr(A) = \sum a_{i,i}$. We note that $tr(AB) = tr(BA)$. This implies that if A is a matrix with respect to some basis, then $tr(A)$ is independent of the choice of basis.*

Let $G = GL_n(\mathbb{C})$ and $\mathtt{g} = \mathtt{gl}_n(\mathbb{C})$, we define a map $\phi : G \longrightarrow Aut(\mathtt{g})$ as $\phi_g(x) = gxg^{-1}$, $g \in G$ and $x \in \mathtt{g}$. ϕ is called adjoint representation or adjoint action. A map ϕ is said to be nilpotent if there exists an integer $r > 0$ such that $\phi^r = 0$.

Lemma 4.6. *[33] Let $x \in \mathtt{g}$. If the linear map $x : V \longrightarrow V$ is nilpotent, then the $adx : \mathtt{g} \longrightarrow \mathtt{g}$ is also nilpotent.*

Remark 4.7. *We once again like to remind ourselves that we are are working strictly on $GL_n(\mathbb{C})$ and $\mathtt{gl}_n(\mathbb{C})$, and $\mathtt{gl}_n(\mathbb{C})$ being a classical Lie algebra, this is equivalent to x being nilpotent in the sense of matrices. $GL_n(\mathbb{C})$ is a connected Lie group with a reductive Lie algebra $\mathtt{gl}_n(\mathbb{C})$.*

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5 Grassmannians and Flag Varieties

In this section, we discuss the connection between Grassmanians and flag varieties. We see this as a necessity to our work because flag varieties are seen as subvarieties of product of Grassmannians.

5.1 The Grassmannians

Let V be an $n-$dimensional vector space over $\mathbb{K}$. For a positive integer $d \leq n$, the Grassmannian $Gr(d,n)$ is defined to be the set of all $d-$dimensional subspace of V.

Example 5.1. *Let $d = 1$, $Gr(1,n) = \mathbb{P}^{n-1}$ the projective space (set of lines through the origin).*
$\mathbb{P}^{n-1} = \{(x_1, \cdots, x_n),\ x_i \neq 0\}\backslash(x_1, \cdots x_n) \sim \lambda(x_1, \cdots, x_n)$
$= \{[x_1 : x_2 : \cdots : x_n]\},\ \lambda \in \mathbb{K}^*$.

In general,suppose A is a $d \times n$ matrix with rank d, then $Gr(d,n) = \{d \times n$ matrix with rank $d\}\backslash$row operation$= GL_n(\mathbb{K})\backslash M_*^{(d,n)}$.Where $GL_n(\mathbb{K})$ is the group of invertible matrices over $\mathbb{K}$ and $M_*^{(d,n)}$ is the set of $d \times n$ matrices over $\mathbb{K}$ with rank d. The map $Gr(d,n) \hookrightarrow \mathbb{P}^{n-1}$, defined by $A \mapsto [Y_{1,2}, Y_{1,3}. \cdots Y_{n-1,n}] = P \in \mathbb{P}^{n-1}$, where $Y_{i,j},\ 1 \leq i < j \leq n$ are the $\binom{n}{d}$ minors for A in $Gr(d,n)$, is called the **Plucker embedding**. We call $Y_{i,j}$ projective coordinates on $Gr(d,n)$.

Theorem 5.2. *[34] The Plucker map is injective .*

Remark 5.3. *It is shown in [34] that $Gr(d,n)$ is precisely the zero set of the well known Plucker relations and hence they are projective varieties.*

5.2 Flag Varieties

Let V be an $n-$dimensional vector space over $\mathbb{C}$, by a flag, we mean a sequence of subspaces $V_i, i = 1, \cdots, n$ ordered by inclusion
$V_\bullet : V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n$ such that $\dim V_i = i$. The collection of all such flags is called the full flag varieties (Whose points are complete flags)which we denote by $\mathcal{F}\ell_n(\mathbb{C})$.

Example 5.4. *Let $\{e_1, e_2, \cdots, e_n\}$ be the standard basis for V, and span$\{e_1, e_2, \cdots, e_i\} = V_i$, then (the standard flag) is given by*
$V_\bullet = \{0\} \subset span\ \{e_1\} \subset span\ \{e_1, e_2\} \subset \cdots \subset span\{e_1, \cdots e_n\}$

Remark 5.5. *Any $g \in GL_n(\mathbb{C})$ can be seen as a flag if we express V_i as the span of the first i columns of g*

5.3 Flag Variety as a Homogeneous Spaces

A homogeneous space X is a topological space which admits a transitive group (G) action. Here, $G = GL_n(\mathbb{C})$ and $X = \mathcal{F}\ell_n(\mathbb{C})$.
The general linear group $GL_n(C)$ acts transitively on $\mathcal{F}\ell_n(\mathbb{C})$. The implication of this is that, for any $V_\bullet \in \mathcal{F}\ell_n(\mathbb{C})$ there exists $g \in GL_n(\mathbb{C})$ which maps $V_\bullet = \{0\} \subset \text{span } \{e_1\} \subset \text{span } \{e_1, e_2\} \subset \cdots \subset \text{span}\{e_1, \cdots e_n\}$ to $V'_\bullet = \{0\} \subset g\text{span } \{e_1\} \subset g\text{span } \{e_1, e_2\} \subset \cdots \subset\ g\text{span}\{e_1, \cdots e_n\}$. $\mathcal{F}\ell_n(\mathbb{C})$ can be identified with elements of $GL_n(\mathbb{C})$ and U_n modulo their respective subgroup which stabilizes the flags.
This simply implies that $\mathcal{F}\ell_n(\mathbb{C})$ can be written as

$$\mathcal{F}\ell_n(\mathbb{C}) = GL_n(\mathbb{C})/B \cong U_n/T^n. \tag{5.1}$$

Hence, $\mathcal{F}\ell_n(\mathbb{C})$ is called a homogeneous space and is of dimension $\frac{n(n-1)}{2}$

Remark 5.6. *This relation between G/B and $\mathcal{F}\ell_n(\mathbb{C})$ gives $\mathcal{F}\ell_n(\mathbb{C})$ a variety structure. There is an obvious embedding $\mathcal{F}\ell_n(\mathbb{C}) \hookrightarrow \prod_{d=1}^{n-1} Gr(d,n)$. The image of the map is a closed subset of this product which is shown in* [89] *to be a projective variety.*

5.4 CW-decomposition of Flag Varieties

It is Possible to obtain a decomposition of $\mathcal{F}\ell_n(\mathbb{C})$ by re-writing any $g \in GL_(\mathbb{C})$ in its column reduced echelon form. In this sense, we say that g is column equivalence to h (i.e $g \sim h$) if h is the column echelon form of g. This relation partitions $\mathcal{F}\ell_n(\mathbb{C})$ into disjoint classes.

If for instance, the echelon form of g is of the form $\begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ where $*$ denote free entries.

It is noted that the leading $1's$ in each column, form a permutation matrix corresponding to $w = 2413 \in S_4$ in one-line notation. This permutation determines the position of $g \in GL_n(\mathbb{C})$ denoted by $\text{pos}(g) = w$. Each $w \in S_n$ determines a cell in $\mathcal{F}\ell_n(\mathbb{C})$ called the **Schubert cell (Bruhat cell)** denoted C_w.

Let $w \in S_n$,the closure $X_w = \overline{C_w} = \sqcup_{v \leq w} C_v$ is called Schubert varieties. Where $\leq$ is a Bruhat order defined on S_n .

Proposition 5.7. *[35] The flag variety is paved by affines $\sqcup_{w \in S_n} C_w$ and has dimension $l(w)$. Where $l(w)$ is the length of w.*

Each Schubert variety X_w is a projective variety such that $dim(X_w) = \ell(w)$. In addition, X_w is an affine space isomorphic to $\mathbb{C}^{\ell(w)}$.

Remark 5.8. *The full flag varieties therefore can be written as a disjoint union of Schubert cells. i.e. $\mathcal{F}\ell_n(\mathbb{C}) = \sqcup_{w \in S_n} C_w$. This decomposition is not just a partition of the flag variety but it forms a CW-decomposition of the flag variety.*

5.5 Cohomology of $\mathcal{F}\ell_n(\mathbb{C})$

The cohomology of $\mathcal{F}\ell_n(\mathbb{C})$ with coefficients in $\mathbb{Z}$ written as

$$H^*(\mathcal{F}\ell_n(\mathbb{C}), \mathbb{Z}) = \bigoplus H^i(\mathcal{F}\ell_n(\mathbb{C}), \mathbb{Z}) \tag{5.2}$$

is a graded ring.

As a module, it is well known that $H^*(\mathcal{F}\ell_n(\mathbb{C}), \mathbb{Z})$ is freely generated by the Schubert class $[X_w]$ indexed by the Weyl group $W \simeq S_n$ of $GL_n(\mathbb{C})$. $H^*(\mathcal{F}\ell_n(\mathbb{C}), \mathbb{Z})$ has a multiplicative structure called cup product. The degree of $[X_w]$ is $2dim[X_w] = 2l(w)$. The k^{th} Betti number of a projective variety Y denoted $b_k(Y) = dimH^k(Y)$. We recall that if Y is a projective variety with a cell decomposition, then the cohomology of Y vanishes in odd degrees, and $H^{2k}(Y, \mathbb{Z})$ is the number of $k-$dimensional cells.

Since $\mathcal{F}\ell_n(\mathbb{C})$ is a non singular complex algebraic variety with $CW-$decomposition, then the cohomology only lives in even dimension. In other words,

$$H^*(\mathcal{F}\ell_n(\mathbb{C})) = \bigoplus H^{2k}, \ k = 0, 1, 2, \cdots .$$

Hence, $b_k((\mathcal{F}\ell_n(\mathbb{C})) = dimH^{2k}((\mathcal{F}\ell_n(\mathbb{C}))$. For instance, let $n = 3$. We recall that $dim(X_w) = l(w)$. Now let $W \simeq S_3 = \{e, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$ Then,

- $X_e = C_e$
- $X_{s_1} = C_e \cup C_{s_1}$
- $X_{s_2} = C_e \cup C_{s_2}$
- $X_{s_1s_2} = C_e \cup C_{s_1} \cup C_{s_2} \cup C_{s_1s_2}$
- $X_{s_2s_1} = C_e \cup C_{s_1} \cup C_{s_2} \cup C_{s_2s_1}$
- $X_{s_1s_2s_1} = C_e \cup C_{s_1} \cup C_{s_2} \cup C_{s_1s_2} \cup C_{s_1s_2s_1}$

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With $dim(X_e) = 0,\ dim(X_{s_1}) = 1 = dim(X_{s_2}),\ dim(X_{s_1s_2}) = 2 = dim(X_{s_2s_1})$ and $dim(X_{s_1s_2s_1}) = 3$ We also recall that $b_k = dimH^{2k}(\mathcal{F}\ell_n(\mathbb{C})$ = number of $k-$dimensional cells.Hence

- $b_0 = dimH^0(\mathcal{F}(3)) = 1$
- $b_1 = dimH^2(\mathcal{F}(3)) = 2$
- $b_2 = dimH^4(\mathcal{F}(3)) = 2$
- $b_3 = dimH^6(\mathcal{F}(3)) = 1$

Therefore, the Poincare polynomial $p_k(t) = \sum_{k=0} b_k t^k = 1 + 2t + 2t^2 + t^3$.
Here,$p_k(t)$ is palindromic which implies that $\mathcal{F}\ell_n(\mathbb{C})$ is smooth.
Some subvarieties of the full flag variety are: Torus Orbit closure; Richardson Varieties; Schubert Varieties and the family of Hessenberg Varieties among which are Peterson and the Springer Varieties.

6 Classification of Nilpotent Orbits in Type A

In this section we consider the classification of nilpotent orbits in $\mathtt{gl}_n(\mathbb{C})$ under the action of $GL_n(\mathbb{C})$. This we do in the framework of partition of integers and Young diagram. Given $r > 0,\ \alpha \in \mathbb{R}$ we denote $J_r(\alpha)$ an $r \times r$ matrix of the form

$$J_r(\alpha) = \begin{pmatrix} \alpha & 1 & 0 & \cdots & 0 \\ 0 & \alpha & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & 1 \\ 0 & 0 & \cdots & 0 & \alpha \end{pmatrix} \tag{6.1}$$

Where α are the eigenvalues and they appear at the main diagonal, 1 appears at the super-diagonal and zero elsewhere. $J_r(\alpha)$ is called Jordan block. We recall that nilpotent matrices in $\mathtt{gl}_n(\mathbb{C})$ have all eigenvalues equal to zero. In view of this, $J_r(\alpha)$ in 6.1 becomes

$$J_r(0) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{6.2}$$

Let $A \in \mathtt{gl}_n(\mathbb{C})$ be nilpotent operator on $\mathbb{C}^n$. We understand from linear algebra that there exists a basis for $\mathbb{C}^n$ such that the matrix A with respect to this basis is of the form

$$X_\lambda = \begin{pmatrix} J_{\lambda_1}(0) & & 0 & \\ & J_{\lambda_2}(0) & & \\ & & \ddots & \\ & 0 & & J_{\lambda_s}(0) \end{pmatrix} \tag{6.3}$$

where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_s)$ and $\sum_{i=1}^{s} \lambda_i = n$.

Remark 6.1. *Each $J_{\lambda_i}(0)$ is an elementary $\lambda_i \times \lambda_i$ Jordan block of type λ_i. Hence, we say that X_λ is of Jordan type λ. X_λ is a nilpotent endomorphism of $\mathbb{C}^{\lambda_1+\lambda_2+\cdots+\lambda_s} = \mathbb{C}^n$*

Theorem 6.2. *(Jordan form)*
Any $n \times n$ nilpotent matrix is similar to precisely one matrix X_λ.

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Example 6.3. *Let $n = 4$, every 4×4 nilpotent matrix has one of the Jordan form below*

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 \\ 0 & 0 & 0 & \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} & \mathbf{1} & 0 & 0 \\ \mathbf{0} & \mathbf{0} & 0 & 0 \\ 0 & 0 & \mathbf{0} & \mathbf{1} \\ 0 & 0 & \mathbf{0} & \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} & \mathbf{1} & 0 & 0 \\ \mathbf{0} & \mathbf{0} & 0 & 0 \\ 0 & 0 & \mathbf{0} & 0 \\ 0 & 0 & 0 & \mathbf{0} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which are parametrized in an obvious way by partition and hence of Jordan types $\lambda = (4) : (3, 1) : (2, 2) : (2, 1, 1)$ and $(1, 1, 1, 1)$ respectively.

The nilpotent cone of the Lie algebra $\mathtt{gl}_n(\mathbb{C})$, denoted by $\mathcal{N}$, consists of all nilpotent elements in $\mathtt{gl}_n(\mathbb{C})$. The general linear group acts on $\mathcal{N}$ by conjugation, this leads to an equivalence relation on $\mathcal{N}$ which partition $\mathcal{N}$ into disjoint classes. The orbits of the action of G on $\mathcal{N}$ for each $X_\lambda \in \mathcal{N}$, corresponding to $\lambda \in \mathcal{P}_n$ denoted by $\mathcal{O}_{X_\lambda}$, are the nilpotent orbit which contains nilpotent matrices of Jordan type λ.

$$i.e, \mathcal{O}_{X_\lambda} = \{B = AX_\lambda A^{-1} : A \in \mathtt{gl}_n(\mathbb{C}\}.$$

Proposition 6.4. *[36] There is one-to-one correspondence between the set of nilpotent orbits of the Lie algebra $\mathtt{gl}_n(\mathbb{C})$ and the set $\mathcal{P}_n$ of partitions of n. This correspondence sends a nilpotent elements X to the partition determined by the block sizes in its Jordan canonical form.The zero orbits corresponds to the partition (1^n). In particular, the set of nilpotent orbits is finite.*

Proposition 6.5. *[36] If A is nilpotent and $c \neq o$ then, cA is also nilpotent. A and cA are conjugate.*

Remark 6.6. *Following the above , it is obvious that every nilpotent orbit in $\mathtt{gl}_n(\mathbb{C})$ corresponds to a unique partition of n, that is, the number of nilpotent orbits in $\mathtt{gl}_n(\mathbb{C})$ is at least $|\mathcal{P}_n|$, in other words, nilpotents orbits are disjoint by the uniqueness of the Jordan normal form. This gives the classification of nilpotent orbits in type A. $\mathcal{N}$ is called nilpotent cone, since being nilpotent is preserved by scaling.*

7 Resolution of Singularities of $\overline{\mathcal{O}}_{X_\lambda}$

Now that we have been able to convince ourselves in the section above that for each $\lambda \in \mathcal{P}_n$, there exists a unique nilpotent orbit denoted by $\mathcal{O}_{X_\lambda}$. In this section, we shall describe the resolution of the singularities of the closure of the algebraic variety $\mathcal{O}_{X_\lambda}$ denoted by $\overline{\mathcal{O}_{X_\lambda}}$. Suppose $x \in \mathtt{gl}_n(\mathbb{C})$ be such that there is $r \in \mathbb{Z}^+$ and $x^r = 0$. Let x acts on $\mathbb{C}^n$, then, we have have a filtration

$$0 = ker(x^0) \subseteq ker(x) \subseteq ker(x^2) \subseteq \cdots \subseteq ker(x^n) = \mathbb{C}^n. \tag{7.1}$$

It is noted that all $g \in B$ satisfies the condition in the equation below.

$$B = \{g \in GL_n(\mathbb{C}) : gker(x^i) \subset ker(x^i)\} \tag{7.2}$$

and

$$\mathtt{b} = \{a \in \mathtt{gl}_n(\mathbb{C}) : a(ker(x^i)) \subset ker(x^i)\}. \tag{7.3}$$

Where $\mathtt{b}$ is the corresponding Lie algebra of B. For a Borel subgroup $B \subset G$, the nilradical $\mathtt{n_b}$ of the Lie algebra of B is the nilpotent cone of $\mathtt{b}$ which we define as;

$$\mathtt{n_b} = \{u \in \mathtt{gl}_n(\mathbb{C}) : u(ker(x^i)) \subset ker(x^{i-1})\}. \tag{7.4}$$

$\mathtt{n_b}$ consists exactly of the nilpotent elements in $\mathtt{b}$. Given a $x \in \mathcal{O}_{X_\lambda}$, the lemma below gives the dimension of the space of $ker(x^i)$.

Lemma 7.1. *[33] If $\lambda = (\lambda_1, \cdots, \lambda_k)$ and $\mu = \lambda'$ is the conjugate of λ with $\mu = (\mu_1, \cdots, \mu_l)$. Then $0 \le s \le l$, $dim(ker(x^s)) = \sum_{i=1}^{s} \mu_i$ and for $s > l$ $dim(ker(x^s)) = n$.*

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Given $G = GL_n(\mathbb{C})$ and B its Borel subgroup. Let $G \times_B \mathtt{n_b}$ be the space obtained as the quotient of $G \times \mathtt{n_b}$ by the right action of B given by $(g, y)b = (gb, b^{-1}y), b \in B, g \in G, y \in \mathtt{n_b}$ Recall that the quotient G/B is a flag variety. In our settings, we identify $gB \in G/B$ with $\{0 \subset g(ker(x)) \subset g(ker(x^2)) \subset \cdots \subset g(ker(x^l))\}$

$$G/B = \{0 \subset V_1 \subset V_2 \subset \cdots \subset V_l = \mathbb{C}^n : dim(V_i) = dim(ker(x^i))\}$$

where V is an l dimensional vector space. G/B is a projective variety, since it can be embedded into product of the Grassmanians $\prod_{i=1}^{n-1} Gr(i, n)$. as stated earlier.

Proposition 7.2. *[33] Define a map $\phi : G \times_B \mathtt{n_b} \longrightarrow G/B \times \mathtt{g}$ by $\phi(g, y) = (gB, gyg^{-1})$. Then the map ϕ is well defined and injective, hence giving $\phi : G \times_B \mathtt{n_b}$ the structure of an algebraic variety (as a closed subvariety of $G/B \times \mathtt{g}$). Furthermore, $im(\phi) = \{((V_i), y) : yV_i \subset V_{i-1}\}$*

Corollary 7.3. *[33] $G \times_B \mathtt{n_b}$ is a vector bundle over G/B, with fibers isomorphic to $\mathtt{n_b}$*

Proof. $G \times_B \mathtt{n_b}$ is identified with the variety $\{((V_i), y) : yV_i \subset V_{i-1}\}$. Suppose f if a projection onto G/B from $G \times_B \mathtt{n_b}$, f is surjective. Fix $gB = (V_i) \in G/B$, the fibre $f^{-1}(V_i) = \{y \in \mathtt{g} : yV_i \subset V_{i-1}\}$. $f^{-1}(V_i)$ is a conjugate of $\mathtt{n_b}$ and hence isomorphic to $\mathtt{n_b}$ as a vector space. □

The map $\pi : G \times_B \mathtt{n_b} \longrightarrow \mathtt{g}$ define by $\pi(g, y) = gyg^{-1}$ is well defined. If $G \times_B \mathtt{n_b}$ is considered as the set $\{((V_i), y) : yV_i \subset V_{i-1}\}$, the map π is projection ($\pi : G \times_B \mathtt{n_b} \longrightarrow \mathtt{g}$).

Corollary 7.4. *[33] The map π is a Projective morphism of varieties .*

Theorem 7.5. *[33] The map $\pi : G \times_B \mathtt{n_b} \longrightarrow \mathtt{g}$ is a resolution of singularities for the orbit closure $\mathcal{O}_{X_\lambda}$. Equivalently the following four statements are true:*

i) The image of π in $\mathtt{g}$ is precisely the orbit closure of $\overline{\mathcal{O}_{X_\lambda}}$.

ii) π is injective when restricted to $\pi^{-1}(\mathcal{O}_{X_\lambda})$. When π^{-1} is restricted to $\mathcal{O}_{X_\lambda}$, $\pi^{-1} : \mathcal{O}_{X_\lambda} \longrightarrow G \times_B \mathtt{n_b}$ is a morphism of algebraic varieties.

iii) As an algebraic variety, $G \times_B \mathtt{n_b}$ is smooth and irreducible.

iv) π is a proper morphism of algebraic varieties .

7.1 Springer Varieties in Type A

Now that we have created a leveled plane ground for our object of study, we shall hence explain what Springer fibers in type A is.
For any $X \in \mathtt{b} \subset \mathcal{N}$ or $\mathcal{O}_{X_\lambda} \subset \mathcal{N}$ the fibers over X i.e $\pi^{-1}(X)$ is called the Springer fibers. This we denote by Spr_λ

Given $X \in \mathcal{O}_{X_\lambda}$, we define the springer fibers

$$Spr_\lambda = \pi^{-1}(X) = \{0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n : XV_i \subset V_{i-1}\}.$$

Geometrically, we define

$$Spr_\lambda = \{0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n : XV_i \subset V_i\}.$$

Remark 7.6. *For any $X, X' \in \mathcal{O}_{X_\lambda}$, the springer fibers Spr_λ and $Spr_{X'}$ are canonically isomorphic. Hence, saying that the Springer fibers over a nilipotent orbit also means the Springer fibers over a nilpotent element in the orbit.*

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7.2 Components of the Springer Fibers

The Springer components is not irreducible in general and the geometry of its irreducible components has been an important topic for current researches. We shall therefore in this section discuss the irreducible components of or dear Springer fibers using standard tableaux.
Let $\mathcal{Y}_\lambda$ be the young diagram corresponding to a partition $\lambda \in \mathcal{P}_n$ and St^λ denotes the set of standard tableaux of shape λ. By standard tableaux, we mean a filling of the Young diagram $\mathcal{Y}_\lambda$ with entries from $[n]$ such that the entries strictly increase from top to bottom, left to right. We refer readers to [10] for details on Young tableaux.
Again, we let $X_\lambda \in \mathcal{N}$ be of Jordan type as explained earlier. The dimension of Spr_λ depends on the diagram $\mathcal{Y}_\lambda$ as shown by the theorem below.

Theorem 7.7. *[14] The variety Spr_λ is equidimensional. Moreover, denote by $\lambda'_1, \lambda'_2, \cdots, \lambda'_r$ the lengths of columns of $\mathcal{Y}$, we have*

$$dim Spr_\lambda = \sum_{q=1}^{r} \frac{\lambda'_q(\lambda'_q - 1)}{2}$$

For any tableaux $T \in St^\lambda$, let $T_i,\ i \leq n$ be the tableau obtained by deleting boxes with numbers $i+1, \cdots, n$. The shape of the tableau T_i is a sub-diagram denoted by $\mathcal{Y}_{T_i} \subset \mathcal{Y}_\lambda$ with i boxes. the standard tableau can be written as the maximal chain of sub-diagrams
$\phi = \mathcal{Y}_{T_0} \subset \mathcal{Y}_{T_1} \subset \cdots \subset \mathcal{Y}_{T_n} = \mathcal{Y}_\lambda$
Let $V_\bullet \in Spr_\lambda$, the restriction map $X/V_i : V_i \longrightarrow V_i$ is a nilpotent endomorphism. Take $\mathcal{Y}_i(V_\bullet) = \mathcal{Y}(X/V_i)$ as the Young diagram corresponding to the Jordan form of $J(X/V_i)$. $J(X/V_i)$ is a partition of i, and$(J(X/V_i))_{i=1}^n$ forms an increasing sequence.
Clearly $\mathcal{Y}_i$ differs from $\mathcal{Y}_{i+1}$ by a corner box. So we obtain a chain of increasing sequence of Young diagrams $\mathcal{Y}_0(V_\bullet), \mathcal{Y}_1(V_\bullet), \cdots, \mathcal{Y}_n(V_\bullet)$ which is equivalent to standard tableau T . Let

$$Spr_{X^T} = \{V_\bullet \in Spr_\lambda : \mathcal{Y}_i(V_\bullet) = \mathcal{Y}_{T_i}, \forall i\}.$$

Theorem 7.8. *[17] For each $T \in St^\lambda$ the subset Spr_{X^T} is a smooth irreducible sub-variety of Spr_λ. Every component of Spr_λ is obtained this way.*

The set Spr_{X^T} for each $T \in St^\lambda$ form a partition of Spr_λ . Hence, we get a decomposition $Spr_X = \sqcup_{T \in St^\lambda} Spr_{X^T}$ parametrized by the standard tableaux of shape λ. For each T, the set Spr_{X^T} is a locally closed, irreducible subset of Spr_X and $dim(Spr_{X^T}) = dim(Spr_X)$

Example 7.9. *For $\lambda = 2, 2, 1$ Spr_X has five irreducible components parametrized by the standard tableaux*

1	2
3	4
5	

1	4
2	5
3	

1	2
3	5
4	

1	5
2	3
4	

and

1	4
2	5
3	

Corollary 7.10. *Let $d = dim(Spr_X)$. For $m \geq 0$, the Betti number $b_m := dim H^{2m}(Spr_X, \mathbb{Q})$ is the number of row-standard tableaux T of shape λ such that $(inv)_\tau = d - m$ [14]*

8 Fixed Points of the Springer Fibers in Type A

Springer fibers associated to a nilpotent operator on $\mathbb{C}^n$ of Jordan type λ admits a natural action of the $k-$dimensional subtorus S^k where k gives the number of Jordan blocks (or equivalently, the number of parts in a partition λ). We stated ealier that the $n-$dimensional torus T^n naturally acts on the flag variety $\mathcal{F}\ell_n(\mathbb{C})$ with the fixed points $(\mathcal{F}\ell_n(\mathbb{C}))^{T^n} \cong S_n$, but T^n does not preserve Spr_λ in general.Hence the introduction of $k-$dimensional torus S^k

$$T^n = \left\{ \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n \end{pmatrix} \in GL_n(\mathbb{C}) : t_i \in \mathbb{C}^*, \ |t_i| = 1, \ i = 1, 2, \cdots, n \right\}$$

And

$$S^k = \left\{ \begin{pmatrix} t_1 N_{\lambda_1} & & & \\ & t_2 N_{\lambda_2} & & \\ & & \ddots & \\ & & & t_k N_{\lambda_k} \end{pmatrix} \in T^n : t_i \in \mathbb{C}^* \ |t_i| = 1, \ i = 1, 2, \cdots, n \right\}$$

Where each N_i is an $i \times i$ regular nilpotent matrix. For a Springer fiber associated to a nilpotent operator, the S^k-fixed points denoted $(Spr_\lambda)^{S^k}$ are in bijective correspondence with the set of row strict tableaux of shape λ, as we now describe.
Let $w = w_1, w_2, \cdots, w_n$ be an element of S_n in one-line notation. For each $\tau \in (rst)_\lambda$ we associate a permutation $w^\tau \in S_n$ by reading the entries of a row-strict tableau τ from left to right, top to bottom. The inverse of w^τ (i.e $(w^\tau)^{-1}$) is a fixed point of Spr_λ. All other fixed points of Spr_λ are written this way.

Example 8.1. *Let $n = 5$ with $\lambda = (2, 2, 1)$ and $\tau = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 2 & 5 \\ \hline 1 & \\ \cline{1-1}\end{array}$, the associated $w^\tau \in S_n$ is $w^\tau = 34251$ by taking its inverse, we have $(w^\tau)^{-1} = 53124$.*

Remark 8.2. *$(w^\tau)^{-1}$ a fixed points of Spr_λ and every other fixed points of Spr_λ are obtained this way. Hence*

$$(Spr_\lambda)^{S^k} = \{(w^\tau)^{-1} : w^\tau \in S_n, \ \tau \in (rst)^\lambda\}.$$

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