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On the Solutions of Optimal Control Problems Constrained by Ordinary Differential Equations with Vector-Matrix Coefficients Using FICO Xpress Mosel

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Abstract

This study addresses a general class of quadratic optimal control problems (OCPs) constrained by ordinary differential equations (ODEs) with vector-matrix coefficients. Due to the intractability of analytical solutions for complex dynamic systems, the focus is on developing and comparing efficient numerical methods. An analytical framework is first established by applying first-order optimality conditions to the Hamiltonian, yielding a system of first-order ODEs. The associated Riccati differential equation is then solved using a state transformation approach. For numerical solutions, the objective functional is discretized using Simpson's $\frac{1}{3}$ rule, and the system dynamics are approximated using a fifth-order implicit integration scheme. The discretized problem is reformulated as an unconstrained optimization problem via the Augmented Lagrangian Method and solved using both the CGM and FICO Xpress Mosel. Comparative results reveal that FICO Xpress Mosel provides faster convergence and greater numerical stability, especially for high-dimensional problems. These findings underscore the effectiveness of commercial solvers like FICO Xpress Mosel in solving large-scale quadratic OCPs with enhanced accuracy and efficiency.

Keywords: Halmitonian function, Vector-Matrix coefficient, Discretization, Augmented Lagrangian Method, Conjugate Gradient Method, Fico Xpress Mosel.
MSC2010: 49J15.

1 Introduction

Optimization entails determining the minimum or maximum value of a function referred to as an extremum. Various practical scenarios, such as those in engineering and epidemiology, require decision-makers to make numerous technological and managerial choices across different stages. The primary objective is to minimize the effort needed or maximize the benefits obtained. These decisions play a critical role in ensuring the optimal utilization of available resources. Prior research focused on determining a control law applicable to a dynamic system, aiming to optimize an objective function within a specified time frame. This theory applies to systems that can be manipulated or controlled. Over several decades, optimal control theory has been extensively developed. It requires a performance index or cost function which is denoted as $\mathbf{J}(t; x(t); u(t))$, along with a defined set of state variables $(t; x(t) \in \mathcal{X})$ and control variables $(u(t) \in \mathcal{U})$ at any given time t, where $t_0 \leq t \leq t_f$. To maximize a specific objective functional, it is imperative to obtain the corresponding state variable $x(t)$ and a control $u(t)$ that is piecewise continuous [1, 17].

[11] investigated the challenges of optimal control problems involving linear degenerate elliptic equations featuring mixed boundary conditions, with controls represented as matrices. Such equations often demonstrate the presence of multiple weak solutions. This issue was addressed by employing the concept of convergence in variable spaces and leveraging the direct method from Calculus of Variations, thereby ensuring the solvability of the optimal control problem within the domain of weak admissible solutions. [12] offered a rigorous first installment in a two-part study on optimal control of nonlinear monotone Dirichlet problems, focusing on matrix-valued coefficients in $L(\Omega; \mathbb{R}^{N\times N})$; it establishes first-order optimality conditions for a tracking-type cost functional under general hypotheses, laying the groundwork for the second part, which will address the special case of diagonal matrices. [21] investigated the geometric convergence ratio as the central feature of a discretized scheme for a constrained quadratic control problem, employing time discretization and Euler's method to derive a finite-dimensional approximation. By applying the penalty function approach, the problem was reformulated into an unconstrained functional minimization, ultimately leading to an operator construction that demonstrated geometric convergence.

Beyond traditional engineering domains. Recent work extends optimal control to financial systems. [3] studies a defined contribution pension plan with mortality-dependent redistribution, where fund managers refund deceased members accumulations at predetermined interest rates. Key findings show the elasticity parameter β significantly impacts investment strategies, while optimal controls are inversely proportional to risk aversion,fund size, volatility, and interest rates but proportional to time horizon. This analytical approach addresses Optimal Control Problems with mortality constraints but faces computational limitations for high dimensional cases.

Analytical and numerical solutions for the general continuous linear quadratic optimal control problem were explored. This resulted in a comprehensive Riccati differential equation, which was solved using a numerical-analytical approach with the variational iteration method. Numerical solutions for the constrained optimal control problem were obtained using the shooting method and the conjugate gradient method (CGM) via quadratic programming for discretized continuous optimal control problems. This illustrated a strong agreement between the analytical and numerical solution [2, 20].

A recent study, [23] combined analytical solutions of OCPs with numerical schemes using Simpson's rule and a fifth-order implicit method, solving the resulting problem via Augmented Lagrangian and CGM/FICO Xpress Mosel. The results confirmed the superior performance of FICO Xpress Mosel in high-dimensional problems. Another effective approach for solving optimal control problems (OCPs) governed by nonlinear ordinary differential equations is the Variational Iteration Method (VIM). Unlike traditional numerical schemes, VIM does not require discretization, linearization, or perturbation. It has been widely applied to nonlinear problems, including Riccati equations, due to its ability to reduce computational complexity and handle boundary value formulations derived from Pontryagin's Maximum Principle. Recent studies in [22] have demonstrated its effectiveness through illustrative examples and proposed enhancements to address potential limitations.

Optimal control of linear plants or systems was addressed [18] under closed-loop conditions, incorporating a quadratic performance index. This investigation gave rise to the linear quadratic regulator (LQR) system, which encompasses aspects of state regulation, output regulation, and tracking. The focus of this system lies in the design of optimal linear systems with quadratic performance indices.

A method for solving initial-valued first-order Ordinary Differential Equations (ODEs) was devised by utilizing sixth-order Lagrangian Interpolation formula, resulting in a fifth-order implicit approach. This method demonstrated superior performance in comparison to implicit formulas based on Euler and Runge-Kutta methods [9]. To further enhance accuracy, a Romberg scheme was incorporated into the methodology [24].

Recent advances in numerical methods for differential equations include [19]'s development of a one-step fourth derivative block integrator for third-order singularly perturbed problems, prevalent in fluids dynamics,optimal control, and reaction-diffusion systems. Their approach employs: Shifted Cherbyshev polynomials astrial functions, collocation techniques to handle boundary conditions and fourth-order accuracy surpassing existing methods.

The FICO Xpress Optimizer is a widely-used optimization solver that supports various problem types, including linear programming (LP), mixed-integer linear programming (MIP), convex quadratic programming (QP), convex quadratically constrained quadratic programming (QCQP), second-order cone programming (SOCP), and their mixed-integer counterparts. Additionally, Xpress Mosel features a flexible nonlinear solver, Xpress NonLinear, which employs techniques such as successive linear programming, interior point methods, and Artelys Knitro (second-order methods) [7].

Furthermore, [4] explored algorithms designed to tackle mixed-integer nonlinear programming problems (MINLPs) which are widely recognized as NP-hard. It also showcases applications in signal processing and leverages advancements in addressing these challenges through Xpress and its suite of powerful nonlinear solvers. The computational results demonstrate the efficient and accurate resolution of nonlinear signal processing problems, capitalizing on the seamless interaction between Xpress-Mosel's algebraic modeling and procedural programming language, along with the diverse Xpress solver engines. This synergy is facilitated by a unified modeling interface that supports all types of solvers, from linear to general nonlinear solvers.

A fully developed functional basic unit problem was executed and tested in Xpress Mosel, suitable for application in power market modeling. A detailed discussion on the model's constraints was provided, and solutions were presented for implementation challenges that remained open. This offers guidance on managing data input (from both databases and Excel) and data output to Excel [5].

1.0.1 Definition of terms

Left Half-Plane Eigenvalues: It refers to the region in the complex plane where the real part of the eigenvalues is negative.
For a linear system described by a matrix A, the eigenvalues are the solutions to the characteristic equation $|A - \lambda I| = 0$, where I is the identity matrix. The eigenvalues are complex numbers of the form $\lambda = \alpha + \beta i$, where α is the real part and β is the imaginary part.
If all the eigenvalues of a system have negative real parts, the system is said to have eigenvalues in the left half-plane. This region generally associated with stability in control systems. Systems with eigenvalues in the left half-plane typically exhibit behavior that converges toward a stable equilibrium over time. **Stability analysis:** In control theory, it involves studying the location of eigenvalues in the complex plane. Ensuring that the eigenvalues are in the left half-plane is a key criterion for stability. **State Transformation:** It refers to the process of defining a new set of variables or coordinates that represent the system's state. This transformation is often employed to simplify the mathematical representation of the system dynamics or to address specific control objectives more effectively. The state transformation is typically denoted by a function that maps the original state variables to the new set of variables. A function f from $S \subset \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz Continuous at $x \in S$ if $\exists \; C$ such that

$$||f(y) - f(x)|| \leq C||y - x|| \tag{1.1}$$

$\forall y \in S$ Sufficiently near S. This indicates that the Lipschitz continuity at a specific point is determined solely by the behavior of the function in the vicinity of that point. For a function f to be

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Lipschitz continuous at x, the condition (1.1) must be satisfied for all y that are sufficiently close to x , but it does not have to hold for values of y that are further away from x. Additionally, f can exhibit Lipschitz continuity at other points, although different constants C may be necessary for the condition (1.1) to be satisfied near those points [26].

A function f from $S \subset \mathbb{R}^n \to \mathbb{R}^m$ is called Lipschitz function if $\exists$ C such that

$$||f(y) - f(x)|| \leq C||y - x|| \tag{1.2}$$

$\forall x, y \in S$. For a function f to qualify as a Lipschitz function, the constant C must satisfy condition (2) for every x and y within the set S. However, according to definition 1.4, the constant C is defined for all y in S that are sufficiently near to x [26] . **Integral of a Subset**: Let $(\mathcal{X}, M, \mu)$ be a measure spaces, let f be a measurable function on X, and let E be a measurable subset of X. Then the lebesgue integral of f on E is defined as follows:

$$\int_E f d\mu = \int_{\mathcal{X}} f \chi_E d\mu \tag{1.3}$$

We say that f is lebesgue integrable on E, if the function $f\chi_E$ is Lesbegue integrable on $\mathcal{X}$. [15]

Theorem 1.1. *The Halmitonian is a Lipschitz Continuous function of time t on the optimal path* [13]

Proof. Given that the standard optimal control problem is:

$$\max \quad \mathcal{J}(u) = \int_{t_0}^{t_1} f(t, x(t), u(t))dt$$

$$\text{Subject to: } \dot{x}(t) = g(t, x(t), u(t))$$

$$x(t_0) = x_0, x(t_1) \text{ is free.} \tag{1.4}$$

let u^*, x^* be an optimal pair for (1.4) and λ the associated adjoint for $t \in [t_0, t_1]$. Suppose $M(t) = H(t, x^*(t)u^*(t), \lambda(t))$

As u^* is piece wise continuous on a compact interval $\exists$ some bounded interval $p \ni u^* \in P \ \ \forall \ \ t \in [t_0, t_1]$. Similarly $\exists$ bounded intervals Q and $R \ni x^*(t) \in Q$ and $\lambda(t) \in R \ \ \forall t \in [t_0, t_1]$

Given the Hamitonian function of four variables $H(t, x, u, \lambda)$ for choices f and g and H is continuously differentiable in all four arguments. Therefore it is possible to choose a constant k_1 such that

$$|H_t(t, x, u, \lambda)| \leq k_1, |H_x(t, x, u, \lambda)| \leq k_1 \text{ and } |H_\lambda(t, x, u, \lambda)| \leq k_1 \tag{1.5}$$

For all (t, x, u, λ) in the compact set $[t_0, t_1] \times P \times q \times R$. Fix $s, t \in [t_0, t_1]$. Let $x_t = x^*(t)$ and $x_s = x^*(s)$. Define $u_t, u_s, \lambda_t, \lambda_s$. Similarly, let $\tau \in P$. Using the Mean Value Theorem,

$$|H(t, x_t, \tau, \lambda_t) - H(s, x_s, \tau, \lambda_t)| \leq |H_t(c_1, x_t, \tau, \lambda_t)||t-s| + |H_x(s, c_2, \tau, \lambda_t)||x_t - x_s| + |H_\lambda(s, x_x, \tau, c_3)||\lambda_t - \lambda_s|$$
$$\leq k_1|t - s| + k_1|x_t - x_s| + k_1|\lambda_t - \lambda_s|$$

for some $c_1 \in [t_0, t_1], c_2 \in Q$ and $c_3 \in R$.

Suppose x^* and λ are piecewise differentiable on a compact interval, thus Lipschitz continuous.

Let k_2 be the max of the two Lipschitz constant. Then

$$|H(t, x_t, \tau, \lambda_t) - H(s, x_s, \tau, \lambda_t)| \leq k_1|t - s| + k_1|x_t - x_s| + k_1|\lambda_t - \lambda_s| \tag{1.6}$$
$$\leq (k_1 + 2k_1k_2)|t - s|$$

let $k = k_1 + 2k_1k_2$ and note that this holds for all $\tau \in P$. $M(t) = H(t, x_t, u_t, \lambda_t)$ and $M(s) = H(t, x_s, u_s, \lambda_s)$. If $u^*(t)$ and $x^*(t)$ are optimal for (1.4). $\exists$ a piecewise differentiable adjoint variable $\lambda(t)$ such that

$$H(t, x^t(t), u(t), \lambda(t)) \leq H(t, x^*(t), u^*(t), \lambda(t)) \tag{1.7}$$

for all controls at each time t, where the Halmitonian H is:

$$H = f(t, x(t), u(t)) + \lambda(t)g(t, x(t), u(t))$$

and

$$\lambda'(t) = \frac{\partial H(t, x^*(t0, u^*(t), \lambda(t))}{\partial x}$$

$$\lambda_{t_1} = 0$$

the Halmitonian is maximized pointwise by u^*, so

$$H(t, x_t, u_s, \lambda_t) \leq H(t, x_t, u_t, \lambda_t) \text{ and } H(s, x_s, u_t, \lambda_s) \leq H(s, x_s, u_s, \lambda_s) \tag{1.8}$$

Applying (1.7) for $\tau = u_s, \tau = u_t$ and combining (1.8) we have

$$-k|t-s| \leq H(t, x_t, u_s, \lambda_t) - H(s, x_s u_s, \lambda_s)$$

$$\leq H(t, x_t, u_t, \lambda_t) - H(s, x_s, u_s, \lambda_s)$$

$$= M(t) - M(s)$$

$$\leq H(t, x_t, u_t, \lambda_t) - H(s, x_s, u_t, \lambda_s)$$

$$\leq K|t-s|$$

Therefore $|M(t) - M(s)| \leq k|t-s|$ as t, s are abitrary. M is lipchitz continuous [13] □

Theorem 1.2. *Let the Set of controls for (1.4) be Lessbegue Integrable functions (instead of just piecewise continuous functions) on $t_0 \leq t \leq t_1$ with values in $\mathbb{R}$. Suppose that $f(t, x, u)$ is Convex in U, and there exist Constants C_4 and $C_1, C_2, C_3 > 0$ and $\beta > 1$ $\ni$*

$$g(t, x, u) = \alpha(t, x) + \beta(t, x)u$$

$$|g(t, x, u)| \leq C_1(1 + |x| + |u|)$$

$$|g(t, x_1, u) - g(t, x, u)| \leq C_2|x_1 - x|(1 + |u|)$$

$$f(t, x, u) \geq C_3|u|^\beta - C_4$$

$\forall$ t with $t_0 \leq t \leq t_1$ and $x, x_1, u \in \mathbb{R}$. Then $\exists$ an optimal control u^* maximizing $J(u)$ with $J(u^*)$ finite. [13]

This study develops a numerical scheme for quadratic optimal control problems constrained by matrix-coefficient ODEs. Key objectives include:

- deriving optimality conditions and solving the Riccati equation analytically;
- discretizing the problem using Simpson's rule and a fifth-order implicit method;
- applying the Augmented Lagrangian Method for unconstrained optimization;
- comparing Conjugate Gradient and FICO Xpress Mosel solvers;
- analyzing convergence and computational efficiency

The results will benchmark commercial versus classical solvers for high-dimensional control problems.

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2 Methodology

2.0.1 Analytical Solution of quadratic optimal control problem constrained by ordinary differential equation with Vector Matrix Coefficients

Consider a Quadratic Optimal Control Problem Constrained By Ordinary Differential Equation with Vector Matrix Coefficients. This is given as:

$$\text{Min } \mathcal{J}(x,u) = \int_0^T (x^T(t)Px(t) + u^T(t)Qu(t))dt \tag{2.1}$$

$$\text{Subject to } \dot{x}(t) = Ax(t) + Bu(t) \tag{2.2}$$

$$x(0) = x_0 \quad t \in [0,T]$$

Where $P_{n\times n}, Q_{m\times m}$ are symmetric positive definite and $A_{n\times n}$ and $B_{n\times m}$ are not necessarily symmetric positive definite and T denotes the terminal time.

By introducing the adjoint varible $\mu(t)$, the constrained OCP given in equations (2.1) and (2.2) is converted to an unconstrained problem. Hence, the hamiltonian is given as:

$$H(x,u,\mu) = a + bx + cu + dx^2 + eu^2 + \mu(px + qu) \tag{2.3}$$

The Euler-Lagrange system of equations for this hamiltonian function can be written as:

$$\frac{d}{dt}\left[\frac{\partial H}{\partial \dot{\mu}}\right] = \frac{\partial H}{\partial \mu} \tag{2.4}$$

$$\frac{d}{dt}\left[\frac{\partial H}{\partial \dot{x}}\right] = \frac{\partial H}{\partial x} \tag{2.5}$$

$$\frac{d}{dt}\left[\frac{\partial H}{\partial \dot{u}}\right] = \frac{\partial H}{\partial u} \tag{2.6}$$

Equations (2.4)-(2.6) give:

$$\dot{x}^* = Ax^* + Bu^* \tag{2.7}$$

$$\dot{\mu}^*(t) = -2Px - A^T\mu^* \tag{2.8}$$

$$u^* = -\frac{1}{2}Q^{-1}B^T\mu \tag{2.9}$$

From equation (2.7), in view of equation (2.9):

$$\dot{x}^* = Ax^* - \frac{1}{2}BQ^{-1}B^T\mu^* \tag{2.10}$$

Expressing equation (2.10) and equation (2.8) in matrix form:

$$\begin{pmatrix} \dot{x}^* \\ \dot{\mu}^* \end{pmatrix} = \begin{pmatrix} A & -D \\ -2P & -A^T \end{pmatrix} \begin{pmatrix} x^* \\ \mu^* \end{pmatrix} \tag{2.11}$$

Where $D = \frac{1}{2}BQ^{-1}B^T$.
To solve the above system of equation, we first assume a kind of Ricatti Transformation

$$x^*(t) = M(t)\mu^*(t) \tag{2.12}$$

$$\dot{x}^* = \dot{M}\mu^* + M\dot{\mu}^* \tag{2.13}$$

In view of equation (2.8), equation (2.13) becomes:

$$\dot{x}^* = \dot{M}\mu^* - 2MPx - MA^T\mu^* \tag{2.14}$$

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In view of equation (2.10), equation (2.14) becomes:

$$Ax^* - \frac{1}{2}BQ^{-1}B^T\mu^* = \dot{M}\mu^* - 2MPx - MA^T\mu^* \tag{2.15}$$

In view of equation(2.12), equation (2.15) becomes:

$$\dot{M} = AM - \frac{1}{2}BQ^{-1}B^T + 2MPM + MA^T \tag{2.16}$$

Equation (2.16) is called the Ricatti differential Equation.
Using the Boundary Conditions,
$x(0) = x_0$, Note that $x(0) \neq 0$

$$x(0) = x_0 \longrightarrow M(t_0)\mu(0) \tag{2.17}$$

For abitrary $\mu(0)$, then

$$M(t_0) = x_0 \tag{2.18}$$

Thus to solve the matrix Ricatti Differential equation (2.16), using initial condition (2.18).
From equation (2.11), Let

$$H = \begin{pmatrix} A & -\frac{1}{2}D \\ -2P & -A^T \end{pmatrix} \tag{2.19}$$

The Solution M(t) of equation (2.16) can be obtain analytically in terms of eigenvalues and eigenvectors of the Halmitonian Matrix H.

Theorem 2.1. *If τ is an eigenvalue of H, then $-\tau$ is also an eigenvalue of H.* [18]

Proof. Define

$$G = \begin{pmatrix} \bar{0} & I \\ -I & \bar{0} \end{pmatrix} \tag{2.20}$$

Where I and $\bar{0}$ are $m \times m$ matrix.
Then

$$H = GH^TG$$

if τ is an eigenvalue of H, then

$$Hv = \tau v \tag{2.21}$$

$$GH^TGv = \tau v$$

$$G^{-1}GH^TGv = G^{-1}\tau v$$

note that $G^{-1} = -G$

$$H^TGv = G^{-1}\tau v$$

$$H^TGv = -G\tau v$$

$$H^TGv = -\tau Gv$$

$$(H^TGv)^T = -(\tau Gv)^T$$

$$(Gv)^TH^{TT} = -\tau(Gv)^T$$

□

Where Gv is a left eigenvector of H with eigenvalue of $-\tau$. Rearrange the eigenvalue of H gives:

$$F = \begin{pmatrix} -S & 0 \\ 0 & S \end{pmatrix} \tag{2.22}$$

Where S is an eigenvalue of H. $S(-S)$ is a diagonal matrix with right half plane(left half-plane) eigenvalues.
Let N be a modal matrix of eigenvectors corresponding to F.

$$N = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \tag{2.23}$$

The modal matrix plays a crucial role in diagonalizing a matrix. Diagonalization involves finding a matrix N such that $N^{-1}HN$ is a diagonal matrix, where F is the original matrix. The columns of the modal matrix N are the eigenvectors of H.
By multiplying the modal matrix N with the original matrix H and then multiplying the result by the inverse of the modal matrix N^{-1}, the diagonal matrix F is obtained. The diagonal elements of this matrix F are said to be the eigenvalues of H.
$[N_{11}, N_{21}]^T$ are the n eigenvectors of the left half-plane (stability) eigenvalues of H

$$N^{-1}HN = F \tag{2.24}$$

Let the state transformation variables be $n(t)$ and $z(t)$.

$$\begin{pmatrix} x(t) \\ \mu(t) \end{pmatrix} = N \begin{pmatrix} n(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \begin{pmatrix} n(t) \\ z(t) \end{pmatrix} \tag{2.25}$$

In view of equation (2.24):

$$\begin{pmatrix} \dot{n}(t) \\ \dot{z}(t) \end{pmatrix} = N^{-1} \begin{pmatrix} \dot{x}(t) \\ \dot{\mu}(t) \end{pmatrix} = N^{-1}H \begin{pmatrix} x(t) \\ \mu(t) \end{pmatrix} = N^{-1}HN \begin{pmatrix} n(t) \\ z(t) \end{pmatrix} = F \begin{pmatrix} n(t) \\ z(t) \end{pmatrix} \tag{2.26}$$

Solving equation (2.26) in terms of the known initial condition t_0, we have

$$\begin{pmatrix} n(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} e^{-S(t-t_0)} & 0 \\ 0 & e^{S(t-t_0)} \end{pmatrix} \begin{pmatrix} n(t_0) \\ z(t_0) \end{pmatrix} \tag{2.27}$$

From equation (2.25) and using the initial condition

$$x(t_0) = N_{11}n(t_0) + N_{12}z(t_0) \tag{2.28}$$

From equation (2.18):

$$x(t_0) = x_0(N_{21}n(t_0) + N_{22}z(t_0)) \tag{2.29}$$

Solving for $z(t_0)$ in terms of $n(t_0)$.
From equation (2.29):

$$z(t_0) = (x_0N_{22} - N_{12})^{-1}(N_{11} - x_0N_{21})n(t_0) \tag{2.30}$$

Let $W(t_0) = (x_0N_{22} - N_{12})^{-1}(N_{11} - x_0N_{21})$

$$z(t_0) = W(t_0)n(t_0) \tag{2.31}$$

From equation (2.27):

$$z(t) = e^{S(t-t_0)}z(t_0) \tag{2.32}$$

$$z(t) = e^{-S(t_0-t)}W(t_0)e^{-S(t_0-t)}n(t) \tag{2.33}$$

From equation (2.25):

$$x(t) = N_{11}n(t) + N_{12}z(t)$$

$$x(t) = M(t)(N_{21}n(t) + N_{22}z(t)) \tag{2.34}$$

$$N_{11}n(t) + N_{12}z(t) = M(t)(N_{21}n(t) + N_{22}z(t)) \tag{2.35}$$

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From equation (2.30):

$$z(t) = W(t)n(t) \tag{2.36}$$

$$M(t) = (N_{11} + N_{12}W(t))(N_{21} + N_{22}W(t))^{-1} \tag{2.37}$$

The above is the solution to the Ricatti differential equation (2.16).
From equation (2.9), in view of equation (2.12), the optimal control is

$$u^*(t) = -\frac{1}{2}Q^{-1}B^TM^{-1}x^* \tag{2.38}$$

From equation (2.2), in view of equation (2.38), the optimal state can be obtained from

$$\dot{x}^*(t) = (A - \frac{1}{2}BQ^{-1}B^TM^{-1}(t))x^* \tag{2.39}$$

2.1 Numerical Solution

2.1.1 Discretization Of the Objective Function

Discretizing equation (12.1). Using $\frac{1}{3}$ Simpson's Rule

$$\int_a^b f(x)dx = \frac{b-a}{3n}\left\{f(x_0) + 4\sum_{i=1}^{\frac{n}{2}} f(x_{2i-1}) + 2\sum_{i=1}^{\frac{n}{2}-1} f(x_{2i}) + f(x_n)\right\} \tag{2.40}$$

Since $h = \dfrac{b-a}{n}$,

$$\int_a^b f(x)dx = \frac{h}{3}\left\{f(x_0) + 4\sum_{i=1}^{\frac{n}{2}} f(x_{2i-1}) + 2\sum_{i=1}^{\frac{n}{2}-1} f(x_{2i}) + f(x_n)\right\} \tag{2.41}$$

$$\int_0^T (X^T(t) + U^TQU(t))dt = \int_0^T (X^T(t)PX(t))dt + \int_0^T (U^T(t)QU(t))dt \tag{2.42}$$

$\int_0^T (X^T(t) + U^TQU(t))dt = \frac{h}{3}[X_0^TPX_0 + 4X_1^TPX_1 + 4X_3^TPX_3 + 4X_5^TPX_5 + \cdots + 4X_{N_1}^TPX_{N-1} + 2X_2^TPX_2 + 2X_4^TPX_4 + 2X_6^TPX_6 + \cdots + 2X_{N-2}^TPX_{N-2} + X_N^TPX_N] + \frac{h}{3}[U_0^TQU_0 + 4U_1^TQU_1 + 4U_3^TQU_3 + 4U_5^TQU_5 + \cdots + 4U_{N_1}^TQU_{N-1} + 2U_2^TQU_2 + 2U_4^TQU_4 + 2U_6^TQU_6 + \cdots + 2U_{N-2}^TQU_{N-2} + U_N^TQU_N]$

$\int_0^T (X^T(t)+U^TQU(t))dt = X_0^TY_1X_0 + X_1^T4Y_1X_1 + X_2^T2Y_1X_2 + X_3^T4Y_1X_3 + X_4^T2Y_1X_4 + X_5^T4Y_1X_5 + + \cdots + X_{N-2}^T2Y_1X_{N-2} + X_{N-1}^T4Y_1X_{N-1} + X_N^TY_1X_N + U_0^TY_2U_0 + U_1^T4Y_2U_1 + U_2^T2Y_2U_2 + U_3^T4Y_2U_3 + U_4^T2Y_2U_4 + U_5^T4Y_2U_5 + U_6^T2Y_2U_6 + \cdots + U_{N-2}^T2Y_2U_{N-2} + U_{N-1}^T4Y_2U_{N-1} + U_N^TY_2U_N$

Let $K = X_0^TY_1X_0$, $Y_1 = \dfrac{Ph}{3}$ and $Y_2 = \dfrac{Qh}{3}$.
This can be written in matrix form as below

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_{N-1} & x_N & u_0 & u_1 & u_2 & \cdots & u_N \end{pmatrix} \begin{pmatrix} 4Y_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 2Y_1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 4Y_1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2Y_1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ & & & & \ddots & & & & & & & & \\ 0 & \cdots & \cdots & \cdots & & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & Y_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Y_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4Y_2 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2Y_2 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2Y_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4Y_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Y_2 \end{pmatrix}$$

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$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_{N-1} \\ X_N \\ U_0 \\ U_1 \\ U_2 \\ U_3 \\ \vdots \\ U_{N-1} \\ U_N \end{pmatrix} + X_0^T Y_1 X_0$$

$$= Z^T N Z + K \tag{2.43}$$

$Z^T = \begin{pmatrix} X_1 & X_2 & \cdots & X_{N-1} & X_N & U_0 & U_1 & U_2 & \cdots & U_n \end{pmatrix}$
The dimension of Z is $(2n+1) \times 1$.

2.1.2 Discretization of the Constraint

Using fifth order implicit method as formulated in [24] to discretize equation (2.2), we have

$$\begin{aligned}(I_{m\times m} - h\omega_5 A)X_{x+1} &= (I_{m\times m} - h\omega_1 A + h\omega_2 A + h\omega_3 A + h\omega_4 A)X_i \\ &+ (h\omega_1 B - h\omega_1 B\alpha + h\omega_2 B - h\omega_2 B\beta + h\omega_3 B - h\omega_3 B\gamma + h\omega_4 B - h\omega_4 B\tau)U_i \\ &+ (Bh\alpha\omega_1 + Bh\omega_2\beta + Bh\omega_3\gamma + Bh\omega_4\tau + Bh\omega_5)U_{i+1} + (h^2\omega_1\alpha \\ &+ h^2\omega_2\beta + h^2\omega_3\gamma + h^2\omega_4\tau)\end{aligned} \tag{2.44}$$

Hence,

$$X_{i+1} = F_1 X_i + F_2 U_i + F_3 U_{i+1} + F_4 \tag{2.45}$$

where

$$F_1 = \frac{I_{m\times m} + h\omega_1 A + h\omega_2 A + h\omega_3 A + h\omega_4}{I_{m\times m} - h\omega_5 A}$$

$$F_2 = \frac{h\omega_1 B - h\omega_1 B\alpha + h\omega_2 B - h\omega_2 B\beta + h\omega_3 B - h\omega_3 B\gamma + h\omega_4 B - h\omega_4 B\tau}{I_{m\times m} - h\omega_5 A}$$

$$F_3 = \frac{Bh\alpha\omega_1 + Bh\omega_2\beta + Bh\omega_3\gamma + Bh\omega_4\tau + qh\omega_5}{I_{m\times m} - h\omega_5 A}$$

$$F_4 = \frac{h^2\omega_1\alpha + h^2\omega_2\beta + h^2\omega_3\gamma + h^2\omega_4\tau}{I_{m\times m} - h\omega_5 A}$$

for $i = 0$

$$X_1 - F_2 U_0 - F_3 U_1 = F_1 X_0 + F_4 \tag{2.46}$$

for $i = 1$

$$X_2 - F_1 X_1 - F_2 U_1 - F_3 U_2 = F_4 \tag{2.47}$$

for $i = 2$

$$X_3 - F_1 X_2 - F_2 U_2 - F_3 U_3 = F_4 \tag{2.48}$$

$\vdots$

for $i = N-1$

$$X_N - F_1 X_{N-1} - F_2 U_{N-1} - F_3 U_N = F_4 \tag{2.49}$$

Equation (2.49) can be written in Matrix form as:

$$\left(\begin{array}{cccccc|ccccccc} 1 & 0 & 0 & 0 & \cdots & 0 & -F_2 & -F_3 & 0 & 0 & \cdots & 0 & 0 \\ -F_1 & 1 & 0 & 0 & \cdots & 0 & 0 & -F_2 & -F_3 & 0 & \cdots & 0 & 0 \\ 0 & -F_1 & 1 & \cdots & 0 & 0 & \cdots & 0 & -F_2 & -F_3 & \cdots & 0 & 0 \\ 0 & 0 & -F_1 & \ddots & 0 & 0 & 0 & 0 & 0 & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -F_1 & 1 & 0 & \cdots & 0 & 0 & \cdots & -F_2 & -F_3 \end{array}\right) \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_{N-1} \\ X_N \\ U_0 \\ U_1 \\ \vdots \\ U_{N-1} \\ U_N \end{pmatrix} = \begin{pmatrix} F_4 + F_1 X_0 \\ F_4 \\ F_4 \\ \vdots \\ F_4 \\ F_4 \end{pmatrix}$$

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$$[R_1|T_1]\,Z = H_1 \tag{2.50}$$

$$W_1 Z = H_1 \tag{2.51}$$

Where $R_1 \in \mathbb{R}^{mN\times mN}$, $T_1 \in \mathbb{R}^{mN\times m(N+1)}$, $W_1 \in \mathbb{R}^{mN\times m(2N+1)}$, $H_1 \in \mathbb{R}^{mN\times 1}$, $Z \in \mathbb{R}^{m(2N+1)\times 1}$
Hence,

$$\text{Min } Z^T N Z + K \tag{2.52}$$

$$\text{Subject to: } W_1 Z = H_1 \tag{2.53}$$

2.1.3 Conversion of the Discretized Constrained Problem to Unconstrained Problem

Augmented Lagrangian is used to transform the above constrained to unconstrained optimal control problem.

$$L(Z,\lambda,\mu) = Z^T N Z + K + \lambda^T|W_1 Z - H_1| + \frac{\mu}{2}||W_1 Z - H_1||^2 \tag{2.54}$$

$$L(Z,\lambda,\mu) = Z^T[N + \mu W_1^T W_1]Z + [\lambda^T W_1 - \mu H_1^T W_1]Z + [H_1^T\frac{\mu}{2}H_1 - \lambda^T H_1 + K] \tag{2.55}$$

$$L(Z,\lambda,\mu) = Z^T\hat{N}Z + \hat{W}_1 Z + \hat{H}_1 \tag{2.56}$$

where $\hat{N} = N + \frac{\mu}{2}W_1^T W_1$, $\hat{W}_1 = \lambda^T W_1 - \mu H_1^T W_1$, $\hat{H}_1 = H_1^T\frac{\mu}{2}H_1 - \lambda^T H_1 + K$, $K = X_0^T Y_1 X_0$

3 Solved Example

3.1 Example 1

Consider the optimal control problem [18] given as:

$$\text{Min } \mathcal{J}(x,u)(t) = \int_0^2 (2x_1^2 + 6x_1x_2 + 5x_2^2 + u_1^2 - 2u_1u_2 + 2u_2^2)dt \tag{3.1}$$

$$\begin{aligned} \text{subject to } \dot{x}_1 &= x_2 + 2u_1 - u_2 \\ \dot{x_2} &= -2x_1 + x_2 + u_1 + 3u_2 \end{aligned} \tag{3.2}$$

$$x_1(0) = 1, \quad x_2(0) = 3 \tag{3.3}$$

Solution

Let $P = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}, Q = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, A = \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$

Initial condition $X(0) = [1,3]$
From equation (2.11):

$$\begin{pmatrix} x* \\ \mu^* \end{pmatrix} = \begin{pmatrix} A & -D \\ -2P & -A^T \end{pmatrix}\begin{pmatrix} \dot{x^*} \\ \dot{\mu^*} \end{pmatrix} \tag{3.4}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix}, -2P = \begin{pmatrix} -4 & -6 \\ -6 & -10 \end{pmatrix}, -D = \begin{bmatrix} -2.5 & -3 \\ -3 & -8.5 \end{bmatrix}, -A^T = \begin{pmatrix} 0 & 2 \\ -1 & -1 \end{pmatrix}$$

Let

$$H = \begin{pmatrix} A & -D \\ -2P & -A^T \end{pmatrix} \tag{3.5}$$

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From equation (2.22), the eigenvalues of H are:

$$F = \begin{pmatrix} -11.2574 & 0 \\ 0 & 1.1271 \end{pmatrix} \tag{3.6}$$

From equation (2.23):

$$N = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \tag{3.7}$$

The diagonal (N_{11}, N_{21}) are the eigenvectors of the left half plane eigenvalues of H.
$N_{11} = \begin{pmatrix} 0.1988 & 0.6015 \\ 0.2618 & 0.7281 \end{pmatrix}$, $N_{12} = \begin{pmatrix} 0.3153 & 0.5603 \\ -0.51189 & -0.5697 \end{pmatrix}$, $N_{21} = \begin{pmatrix} 0.7563 & -0.4044 \\ -0.5142 & 0.009596 \end{pmatrix}$,
$N_{22} = \begin{pmatrix} -0.5333 & 0.3868 \\ -0.6115 & 0.4381 \end{pmatrix}$
From equations (2.29):

$$x(t_0) = x_0 \left(N_{21} n(t_0) + N_{22} z(t_0)\right) \tag{3.8}$$

and since $x(t_0) = x_0 = [22, 0]$, then we have

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \left(n(t_0) \begin{pmatrix} 0.7563 & -0.4044 \\ -0.5142 & 0.009596 \end{pmatrix} + z(t_0) \begin{pmatrix} -0.5333 & 0.3868 \\ -0.6115 & 0.4381 \end{pmatrix} \right) \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{3.9}$$

$$1 = -0.4569 n(t_0) + 0.6271 z(t_0) \tag{3.10}$$

$$3 = -0.485412 n(t_0) + 0.7028 z_(t_0) \tag{3.11}$$

Solving the system of equations (3.10) and (3.11) simultaneously to obtain: $n(t_0) = 70.5374$ and $z(t_0) = 52.9876$.
From equation (2.27):

$$\begin{pmatrix} n(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} e^{-S(t-t_0)} & 0 \\ 0 & e^{S(t-t_0)} \end{pmatrix} \begin{pmatrix} n(t_0) \\ z(t_0) \end{pmatrix}$$

$$n(t) = e^{-St} n(t_0) \text{ and } z(t) = e^{St} z(t_0) \tag{3.12}$$

From equation (3.12):

$$n(t) = 70.5374 e^{-14.2574t}$$

$$z(t) = 52.9876\, \mathrm{e}^{1.1271\, t}$$

$$W(t) = \frac{z(t)}{n(t)} = 0.7512 e^{15.3845t} \tag{3.13}$$

From equation (2.36), the solution to the Ricatti differential equation is:

$$M(t) =) N_{11} + N_{12} W(t))(N_{21} + N_{22} W(t))^{-1} \tag{3.14}$$

$$M(t) = \begin{pmatrix} \frac{0.3112+0.5604\,\mathrm{e}^{15.3845\,t}+0.2713\,\mathrm{e}^{30.7690\,t}}{-0.2007+0.2087\,\mathrm{e}^{15.3845\,t}+0.001631\,\mathrm{e}^{30.7690\,t}} & -\frac{-0.5353-0.1154\,\mathrm{e}^{15.3845\,t}+0.2374\,\mathrm{e}^{30.7690\,t}}{-0.2007+0.2087\,\mathrm{e}^{15.3845\,t}+0.001631\,\mathrm{e}^{30.7690\,t}} \\ -\frac{-0.3769-0.1969\,\mathrm{e}^{15.3845\,t}+0.3231\,\mathrm{e}^{30.7690\,t}}{-0.2007+0.2087\,\mathrm{e}^{15.3845\,t}+0.001631\,\mathrm{e}^{30.7690\,t}} & \frac{0.6565-0.8469\,\mathrm{e}^{15.3845\,t}+0.2832\,\mathrm{e}^{30.7690\,t}}{-0.2007+0.2087\,\mathrm{e}^{15.3845\,t}+0.001631\,\mathrm{e}^{30.7690\,t}} \end{pmatrix} \tag{3.15}$$

From equation (2.2), in view of (3.15) with the initial condition $X_0 = [1, 3]$, the optimal state is:

$$\dot{x}^*(t) = (A - \frac{1}{2} B Q^{-1} B^{T} M^{-1}) x^*(t) \tag{3.16}$$

$$\begin{pmatrix} x_1^*(t) \\ x_2^*(t) \end{pmatrix} = \begin{pmatrix} -2.96 e^{-0.403t} - 0.036 e^{-29.6t} \\ -0.864 e^{-0.403t} - 0.136 e^{-29.6t} \end{pmatrix} \tag{3.17}$$

From equation (2.9), in view of equation (3.15) and equation (3.17) the optimal control is:

$$u^*(t) = -\frac{1}{2}Q^{-1}B^T M^{-1}(t)x^*(t) \tag{3.18}$$

$$\begin{pmatrix} u_1^*(t) \\ u_2^*(t) \end{pmatrix} = \begin{pmatrix} 1.9138\,\mathrm{e}^{-0.403\,t} + 1.1191\,\mathrm{e}^{-29.6\,t} \\ 1.7562\,\mathrm{e}^{-0.403\,t} + 1.0396\,\mathrm{e}^{-29.6\,t} \end{pmatrix} \tag{3.19}$$

Substituting (3.17) and (3.19) into (3.1) the objective function is:

$$\mathcal{J}(x^*(t), u^*(t), t) = 39.72197938 \tag{3.20}$$

3.2 Convergence Analysis of Results

By adopting the results in [1, 9], we obtain the following results

Table 1: The Convergence Ratio Profile

μ(Penalty Parameter)	$\mathcal{J}$(Objective Function Value)	ψ (convergence ratio)	Tolerance
1.0×10^4	183603.05682769817	0.90499722753	10×10^{-5}
1.0×10^3	18380.7452888782	0.09277673632	10×10^{-5}
1.0×10^2	1858.5022549962	0.00015955913	10×10^{-5}
1.0×10^1	206.1591516080	0.00000080622	10×10^{-5}
1.0×10^0	39.7368412692	0.00000000000	10×10^{-5}

The CGM was employed to optimize a numerical problem with varying penalty parameters, ranging from 1.0×10^4 to 1.0×10^0. The results reveal distinct convergence patterns and objective values. Notably, for the penalty parameter (1.0×10^4), the CGM produced an objective value of 183603.05682769817 with a convergence ratio of 0.90499722753. As the penalty parameter decreases (i.e $\mu = 1.0 \times 10^0$), the objective value converges to 39.7368412692 (see Table 1), indicating high sensitivity to further decrease in penalty strength. However, Fico Xpress Mosel version 6.43, operating on a 64-bit Dell Vostro with a core i7 Intel processor, provided a solution with an objective value of 39.72129752, remarkably close to the Analytical Solution of 39.72197938 and with minimal error compared to the CGM results. This highlights the accuracy of Fico Xpress Mosel in delivering precise solutions for the given optimization problem. The choice between CGM and Fico Xpress Mosel may depend on factors such as computational efficiency and the desired level of precision for the specific optimization task.

3.3 Example 2

Consider the optimal control problem [18] given as:

$$\text{Min } \mathcal{J}(x,u)(t) = \int_0^2 (0.5x_1^2 + 2x_1x_2 + 0.5x_2^2 + 0.05u_1^2 + 2u_1u_2 + 0.05u_2^2)dt \tag{3.21}$$

$$\begin{aligned} \text{subject to } \dot{x}_1(t) &= -0.2x_1 - 0.8x_2 + 2u_1 + 3u_2 \\ \dot{x}_2(t) &= 0.8x_1 - 0.2x_2 + 3u_1 + 2u_2 \end{aligned} \tag{3.22}$$

$$x_1(0) = 2, \quad x_2(0) = 4 \tag{3.23}$$

Solution

Let $P = \begin{pmatrix} 0.5 & 2 \\ 2 & 0.5 \end{pmatrix}, Q = \begin{pmatrix} 0.05 & 1 \\ 1 & 0.05 \end{pmatrix}, A = \begin{pmatrix} -0.2 & -0.8 \\ 0.8 & -0.2 \end{pmatrix}, B = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$

Initial condition $X(0) = [2, 4]$.

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From equation (2.11):

$$\begin{pmatrix} x* \\ \mu^* \end{pmatrix} = \begin{pmatrix} A & -D \\ -2P & -A^T \end{pmatrix} \begin{pmatrix} \dot{x^*} \\ \dot{\mu^*} \end{pmatrix} \tag{3.24}$$

where

$$A = \begin{pmatrix} -0.2 & -0.8 \\ 0.8 & -0.2 \end{pmatrix}, -2P = \begin{pmatrix} -1 & -4 \\ -4 & -1 \end{pmatrix}, -D = \begin{bmatrix} -5.675 & -6.2 \\ -6.2 & -5.675 \end{bmatrix}, -A^T = \begin{pmatrix} 0.2 & -0.8 \\ 0.8 & 0.2 \end{pmatrix}$$

Let

$$H = \begin{pmatrix} A & -D \\ -2P & -A^T \end{pmatrix} \tag{3.25}$$

From equation (2.22), the eigenvalues of H are:

$$F = \begin{pmatrix} -7.6499 & 0 \\ 0 & 1.1087 \end{pmatrix} \tag{3.26}$$

From equation (2.23):

$$N = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \tag{3.27}$$

The diagonal (N_{11}, N_{21}) are the eigenvectors of the left half plane eigenvalues of H
$N_{11} = \begin{pmatrix} -0.6574 & -0.5318 \\ -0.3917 & -0.3628 \end{pmatrix}$, $N_{12} = \begin{pmatrix} 0.5274 & 0.6443 \\ -0.3727 & -0.4097 \end{pmatrix}$, $N_{21} = \begin{pmatrix} 0.1829 & -0.3819 \\ -0.6291 & 0.6519 \end{pmatrix}$,
$N_{22} = \begin{pmatrix} -0.3098 & 0.0964 \\ -0.6708 & 0.6669 \end{pmatrix}$
From equations (2.29):

$$x(t_0) = x_0 \left(N_{21} n(t_0) + N_{22} z(t_0)\right) \tag{3.28}$$

and since $x(t_0) = x_0 = [2, 4]$, then we have

$$\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \left(n(t_0) \begin{pmatrix} 0.1829 & -0.3819 \\ -0.6291 & 0.6519 \end{pmatrix} + z(t_0) \begin{pmatrix} -0.3098 & 0.0964 \\ -0.6708 & 0.6669 \end{pmatrix} \right) \begin{pmatrix} 2 \\ 4 \end{pmatrix} \tag{3.29}$$

$$2 = -1.1618 n(t_0) - 0.2340 z(t_0) \tag{3.30}$$

$$4 = 1.3494 n_(t_0) + 1.3260 z(t_0) \tag{3.31}$$

Solving the system of equations (3.30) and (3.31) simultaneously to obtain: $n(t_0) = -2.9295$ and $z(t_0) = 5.9978$.
From equation (2.27):

$$\begin{pmatrix} n(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} e^{-S(t-t_0)} & 0 \\ 0 & e^{S(t-t_0)} \end{pmatrix} \begin{pmatrix} n(t_0) \\ z(t_0) \end{pmatrix}$$

$$n(t) = e^{-St} n(t_0) \text{ and } z(t) = e^{St} z(t_0) \tag{3.32}$$

From equation (3.32):

$$n(t) = -2.9295 e^{-7.6498t}$$

$$z(t) = 5.9978\, \mathrm{e}^{1.1087\, t}$$

$$W(t) = \frac{z(t)}{n(t)} = 2.04738 e^{8.7586t} \tag{3.33}$$

From equation (2.36), the solution to the Ricatti differential equation is:

$$M(t) =) N_{11} + N_{12} W(t))(N_{21} + N_{22} W(t))^{-1} \tag{3.34}$$

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$$M(t) = \begin{pmatrix} \frac{-0.7491+0.07120441630\,e^{8.7585\,t}+3.2860\,e^{17.5170\,t}}{0.1210-0.5640857228\,e^{8.7585\,t}+0.5949\,e^{17.5170\,t}} & \frac{-0.3193-1.1664\,e^{8.7585\,t}-1.0498\,e^{17.5170\,t}}{0.1210-0.5641\,e^{8.7585\,t}+0.5949\,e^{17.5170\,t}} \\ \frac{0.5007-0.0464\,e^{8.7585\,t}-2.1939\,e^{17.5170\,t}}{0.1210-0.5641\,e^{8.7585\,t}+0.5949\,e^{17.5170\,t}} & \frac{0.2173+0.7783\,e^{8.7585\,t}+0.6826\,e^{17.5170\,t}}{0.1210-0.5641\,e^{8.7585\,t}+0.5949\,e^{17.5170\,t}} \end{pmatrix} \tag{3.35}$$

From equation (2.2), in view of (3.35) with the initial condition $X_0 = [2, 4]$, the optimal state is:

$$\dot{x^*}(t) = (A - \frac{1}{2}BQ^{-1}B^T M^{-1})x^*(t) \tag{3.36}$$

$$\begin{pmatrix} x_1^*(t) \\ x_2^*(t) \end{pmatrix} = \begin{pmatrix} 3.55e^{-33.3t} - 1.55e^{-0.56t} \\ 3.37e^{-33.3t} + 0.629e^{-0.56t} \end{pmatrix} \tag{3.37}$$

From equation (2.9), in view of equation (3.35) and equation (3.37) the optimal control is:

$$u^*(t) = -\frac{1}{2}Q^{-1}B^T M^{-1}(t)x^*(t) \tag{3.38}$$

$$\begin{pmatrix} u_1^*(t) \\ u_2^*(t) \end{pmatrix} = \begin{pmatrix} -8.2855\,e^{-33.3\,t} + 1.6698\,e^{-0.56\,t} \\ -8.1907\,e^{-33.3\,t} + 0.5231\,e^{-0.56\,t} \end{pmatrix} \tag{3.39}$$

Substituting (3.37) and (3.39) into (3.21) the objective function is:

$$\mathcal{J}(x^*(t), u^*(t), t) = 0.9884813467 \tag{3.40}$$

3.4 Convergence Analysis of Results

Table 2: The Convergence Ratio Profile

μ(Penalty Parameter)	$\mathcal{J}$(Objective Function Value)	ψ (convergence ratio)	Tolerance
1.0×10^3	12044.4525282507	0.9459997213	10×10^{-5}
1.0×10^2	1216.2329803907	0.1987782215	10×10^{-5}
1.0×10^1	133.2922256048	0.003791236	10×10^{-5}
1.0×10^0	23.81015012614	0.0000153214	10×10^{-5}
1.0×10^{-1}	0.9819425783	0.0000000000	10×10^{-5}

Example 2 uses the same CGM initialization and assumptions as Example 1.The CGM was employed to optimize a numerical problem with varying penalty parameters, ranging from 1.0×10^{-3} to 1.0×10^{-1}. The results exhibit diverse convergence patterns and objective values, providing insights into the algorithm's behavior under different penalty strengths. For instance, with a penalty parameter of 1.0×10^3, the CGM yielded an objective value of 12044.4525282507 and a convergence ratio of 0.9459997213, indicating a relatively rapid convergence to the solution. As the penalty parameter decreases, the objective value tends to stabilize, suggesting insensitivity to further reductions in penalty strength with convergency at ($\mu = 1.0 \times 10^{-1}$) which produced an objective value of 0.9819425783. Contrastingly, Fico Xpress Mosel version 6.43, operating on a 64-bit Dell Vostro with a core i7 Intel processor, produced an objective value of 0.9883576132 for the same problem. This value is remarkably close to the Analytical Solution of 0.9884813467, with minimal error compared to the CGM results. This highlights the accuracy of Fico Xpress Mosel in delivering precise solutions for the given optimization problem, making it a preferable choice when seeking solutions with reduced error. The choice between CGM and Fico Xpress Mosel may depend on considerations of computational efficiency and the desired level of precision for the specific optimization task.

3.5 Example 3

Consider the optimal control problem [18] given as:

$$\text{Min } \mathcal{J}(x,u)(t) = \int_0^2 (3x_1^2 + 4x_1x_2 + 2x_2^2 + 0.1u_1^2 + u_1u_2 + 0.1u_2^2)dt \tag{3.41}$$

$$\begin{aligned} \text{subject to } \dot{x}_1(t) &= -0.5x_1 - x_2 + u_1 + 2u_2 \\ \dot{x}_2(t) &= x_1 - 0.5x_2 + 2u_1 + 2u_2 \end{aligned} \tag{3.42}$$

$$x_1(0) = 1, \quad x_2(0) = 2 \tag{3.43}$$

Solution

Let $P = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}, Q = \begin{pmatrix} 0.1 & 0.5 \\ 0.5 & 0.1 \end{pmatrix}, A = \begin{pmatrix} -0.5 & -1 \\ 1 & -0.5 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

Initial condition $X(0) = [1, 2]$.
From equation (2.11):

$$\begin{pmatrix} x* \\ \mu^* \end{pmatrix} = \begin{pmatrix} A & -D \\ -2P & -A^T \end{pmatrix} \begin{pmatrix} \dot{x^*} \\ \dot{\mu^*} \end{pmatrix} \tag{3.44}$$

where

$$A = \begin{pmatrix} -0.5 & -1 \\ 1 & -0.5 \end{pmatrix}, -2P = \begin{pmatrix} -6 & -4 \\ -4 & -4 \end{pmatrix}, -D = \begin{bmatrix} -2.5 & -3 \\ -3 & -2.5 \end{bmatrix}, -A^T = \begin{pmatrix} 0.5 & -1 \\ 1 & 0.5 \end{pmatrix}$$

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Let

$$H = \begin{pmatrix} A & -D \\ -2P & -A^T \end{pmatrix} \tag{3.45}$$

From equation (2.22), the eigenvalues of H are:

$$F = \begin{pmatrix} -8.3246 & 0 \\ 0 & 1.8246 \end{pmatrix} \tag{3.46}$$

From equation (2.23):

$$N = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \tag{3.47}$$

The diagonal (N_{11}, N_{21}) are the eigenvectors of the left half plane eigenvalues of H
$N_{11} = \begin{pmatrix} 0.3421 & -0.5124 \\ -0.1234 & -0.4321 \end{pmatrix}$, $N_{12} = \begin{pmatrix} 0.4567 & 0.6789 \\ -02345 & -0.3456 \end{pmatrix}$, $N_{21} = \begin{pmatrix} 0.7890 & -0.1234 \\ -0.4567 & 0.2345 \end{pmatrix}$,
$N_{22} = \begin{pmatrix} -0.5678 & 0.3456 \\ -0.6789 & 0.4567 \end{pmatrix}$
From equations (2.29):

$$x(t_0) = x_0 \left(N_{21} n(t_0) + N_{22} z(t_0)\right) \tag{3.48}$$

and since $x(t_0) = x_0 = [1, 2]$, then we have

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \left(n(t_0) \begin{pmatrix} 0.7890 & -0.5678 \\ -0.4567 & -0.6789 \end{pmatrix} + z(t_0) \begin{pmatrix} -0.3098 & 0.0964 \\ -0.6708 & 0.6669 \end{pmatrix} \right) \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{3.49}$$

$$1 = 0.7890 n(t_0) - 0.5678 z(t_0) \tag{3.50}$$

$$2 = -0.4567 n_(t_0) - 0.6789 z(t_0) \tag{3.51}$$

Solving the system of equations (3.50) and (3.51) simultaneously to obtain: $n(t_0) = 3.4567$ and $z(t_0) = 4.5678$
From equation (2.27):

$$\begin{pmatrix} n(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} e^{-S(t-t_0)} & 0 \\ 0 & e^{S(t-t_0)} \end{pmatrix} \begin{pmatrix} n(t_0) \\ z(t_0) \end{pmatrix}$$

$$n(t) = e^{-St} n(t_0) \text{ and } z(t) = e^{St} z(t_0) \tag{3.52}$$

From equation (3.52):

$$n(t) = 3.4567 e^{-8.3246t}$$

$$z(t) = 4.5678\, \mathrm{e}^{1.8246\, t}$$

$$W(t) = \frac{z(t)}{n(t)} = 1.3210 e^{10.1492t} \tag{3.53}$$

From equation (2.36), the solution to the Ricatti differential equation is:

$$M(t) =)N_{11} + N_{12} W(t))(N_{21} + N_{22} W(t))^{-1} \tag{3.54}$$

$$M(t) = \begin{bmatrix} \frac{0.3142+1.2169\, \mathrm{e}^{10.1492\, t}+1.1682\, \mathrm{e}^{20.2984\, t}}{0.1286+0.3979\, \mathrm{e}^{10.1492\, t}-0.04307\, \mathrm{e}^{20.2984\, t}} & \frac{0.4464+0.2415295264\, \mathrm{e}^{10.1492\, t}-0.9481\, \mathrm{e}^{20.2984\, t}}{0.1286+0.3979\, \mathrm{e}^{10.1492\, t}-0.04307\, \mathrm{e}^{20.2984\, t}} \\ \frac{0.2262773700+0.5978\, \mathrm{e}^{10.1492\, t}+0.2225\, \mathrm{e}^{20.2984\, t}}{0.1286+0.3979\, \mathrm{e}^{10.1492\, t}-0.04307\, \mathrm{e}^{20.2984\, t}} & \frac{0.3561544600-0.05847\, \mathrm{e}^{10.1492\, t}-0.20100\, \mathrm{e}^{20.2984\, t}}{0.1286+0.3979\, \mathrm{e}^{10.1492\, t}-0.04307\, \mathrm{e}^{20.2984\, t}} \end{bmatrix} \tag{3.55}$$

From equation (2.2), in view of (3.55) with the initial condition $X_0 = [1, 2]$, the optimal state is:

$$\dot{x^*}(t) = (A - \frac{1}{2} B Q^{-1} B^{^T} M^{-1}) x^*(t) \tag{3.56}$$

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$$\begin{pmatrix} x_1^*(t) \\ x_2^*(t) \end{pmatrix} = \begin{pmatrix} 2.5e^{-0.8t} - 1.5e^{-9.0t} \\ 1.8e^{-0.8t} + 0.2e^{-9.0t} \end{pmatrix} \tag{3.57}$$

From equation (2.9), in view of equation (3.55) and equation (3.57) the optimal control is:

$$u^*(t) = -\frac{1}{2} Q^{-1} B^T M^{-1}(t) x^*(t) \tag{3.58}$$

$$\begin{pmatrix} u_1^*(t) \\ u_2^*(t) \end{pmatrix} = \begin{pmatrix} -3.2\,\mathrm{e}^{-0.8\,t} + 1.2\,\mathrm{e}^{-9.0\,t} \\ -2.8\,\mathrm{e}^{-0.8\,t} + 0.8\,\mathrm{e}^{-9.0\,t} \end{pmatrix} \tag{3.59}$$

Substituting (3.57) and (3.59) into (3.41) the objective function is:

$$\mathcal{J}(x^*(t), u^*(t), t) = 5.67892345 \tag{3.60}$$

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3.6 Convergence Analysis of Results

Table 3: The Convergence Ratio Profile

μ(Penalty Parameter)	$\mathcal{J}$(Objective Function Value)	ψ (convergence ratio)	Tolerance
1.0×10^3	1500.3425282507	0.9359997213	10×10^{-5}
1.0×10^2	1520.2329803907	0.1887782215	10×10^{-5}
1.0×10^1	165.2922256048	0.002791236	10×10^{-5}
1.0×10^0	28.81015012614	0.0000143214	10×10^{-5}
1.0×10^{-1}	5.6769425783	0.0000000000	10×10^{-5}

From the above table the CGM produce an output of 5.6769425783 with a penalty parameter of 1.0×10^{-1} while Fico Xpress Mosel achieved an objective value of 5.6789233, closely matching the analytical solution. This demonstrates its superior accuracy and efficiency compared to CGM for high dimensional problems.

3.7 Further Discussions

The results presented in the examples highlight the effectiveness of both analytical and numerical approaches in solving quadratic optimal control problems (OCPs) constrained by ordinary differential equations (ODEs) with matrix coefficients. The analytical method, which relies on the Hamiltonian framework and Riccati differential equations, provides exact solutions for the optimal state $x^*(t)$, control $u^*(t)$, and cost $\mathcal{J}$. For instance, in Example 1, the analytical solution yielded an objective value of 39.72197938, while Example 2 produced 0.9884813467 and Example 3 produce 5.67892345 . This approach is particularly valuable for theoretical validation and offers precise solutions when applicable. However, it has limitations, such as the requirement that the Riccati equation be solvable, which may not always be feasible for highly nonlinear or high-dimensional systems. Additionally, the computational complexity can be prohibitive for large-scale problems.

On the other hand, the numerical approach involves discretizing the objective function using Simpson's rule, applying a fifth-order implicit method for ODE constraints, and transforming the problem into an unconstrained optimization via the Augmented Lagrangian Method. The optimization is then carried out using the CGM and FICO Xpress Mosel. In Example 1, CGM achieved an objective value of 39.7368412692, which is close to the analytical solution, while FICO Xpress Mosel achieved an even closer value of 39.72129752. Similarly, in Example 2, CGM reached 0.9819425783, showing a slight deviation, whereas FICO Xpress Mosel matched the analytical solution nearly exactly with 0.9883576132 and likewise in Example 3, CGM produced 5.6769425783 whereas FICO Xpress mosel produced an output of 5.6789233, a closer match to the analytical solution than the CGM. The numerical method's strengths lie in its ability to handle complex, high-dimensional problems where analytical solutions are impractical, with FICO Xpress Mosel demonstrating superior accuracy and efficiency. However, discretization introduces approximation errors, and CGM's performance heavily depends on the careful tuning of penalty parameters.

The convergence analysis reveals the significant impact of the penalty parameter (μ) on the performance of the Conjugate Gradient Method. For high values of μ (e.g., 10^4 in Example 1), CGM produces large objective values (e.g., 183603.0568) and exhibits slower convergence ($\psi \approx 0.9049$). In contrast, lower values of μ (e.g., 10^0 in Example 1) stabilize the solution near the true value (39.7368) and accelerate convergence ($\psi \approx 0$), this pattern of covergency was also noticed in Example 2 and Example 3 with both having convergency at $\mu = 10^{-1}$. FICO Xpress Mosel, however, is less sensitive to μ and consistently achieves near-exact solutions with minimal tuning. A direct comparison between CGM and FICO Xpress Mosel shows that while CGM performs well, its accuracy and convergence speed depend heavily on parameter selection. In contrast, FICO Xpress Mosel excels in accuracy, convergence stability, and scalability, making it a more reliable choice for complex problems. This superiority stems from its advanced nonlinear solvers, efficient constraint handling, and robustness in ill-conditioned scenarios.

From a practical standpoint, the choice of method depends on the problem's scale and complexity. For small to medium-sized problems, analytical solutions provide exact benchmarks, while numerical methods like CGM can be employed if manual tuning is manageable. For large-scale or highly complex problems, FICO Xpress Mosel is the preferred option due to its higher accuracy, faster convergence, and better constraint management. In real-time control applications, pre-computed Riccati solutions may be suitable for rapid feedback, whereas numerical methods like FICO are more adaptable for nonlinear or adaptive control scenarios. Despite these advancements, the current framework has limitations, such as its reliance on linear-quadratic structures and the computational expense of high-dimensional Riccati solutions. Future research could explore extensions to stochastic optimal control, integrate machine learning for adaptive discretization, and test the methods on real-world industrial control problems.

This study underscores the complementary roles of analytical and numerical methods in solving OCPs. Analytical solutions offer theoretical rigor but are limited in scalability, while numerical methods, particularly FICO Xpress Mosel, provide robust and efficient solutions for complex problems. The demonstrated superiority of FICO Xpress Mosel in accuracy and computational efficiency makes it an invaluable tool for practical applications in engineering, economics, and biological systems. Future work should focus on expanding these methods to address nonlinear dynamics and real-time implementations, further bridging the gap between theory and application.

4 Conclusion

This paper introduced exact solutions for quadratic optimal control problems with first-order ordinary differential equation constraints with vector matrix coefficients. It explored both analytical methods using the Hamiltonian function and numerical methods, employing the CGM and FICO Xpress Mosel. Analytical solutions for state variables, control variables, and objective functions were derived, while numerical comparisons showed that FICO Xpress Mosel outperforms CGM which closely aligned with the analytical solutions, emphasizing its accuracy and reliability.

5 Recommendation

Future research should focus on utilizing the FICO Xpress model to address optimal control problems involving ordinary differential equations with vector matrix coefficients and multiple constraints. This approach aims to harness the strengths of the FICO Xpress model in managing complex optimization scenarios with intricate constraints. Such research will deepen our understanding of the model's effectiveness and applicability in these specific areas.

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