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Optimal Investment strategy with Currency Uncertainty

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Abstract

Currency uncertainty, characterized by fluctuations in exchange rates, significantly impacts the valuation of investments and the timing of investment decisions. This paper develops a real options framework incorporating a novel currency uncertainty ratio to capture the impact of exchange rate fluctuations on investment timing. We analyze two cases: (i) project value following a geometric Brownian motion (GBM) with a constant investment cost and (ii) both project value and investment cost evolving as GBMs. Using stochastic control and optimal stopping theory, we derive closed-form solutions for the investment thresholds and option values. We further conduct a sensitivity analysis to explore how investment thresholds respond to changes in currency risk, volatility, and discount rates. Our results suggest that currency uncertainty significantly influences investment timing, often delaying investment decisions. These findings have important implications for multinational corporations, policymakers, and financial analysts concerned with investment under foreign exchange risk.

Keywords: Currency Uncertainty, Optimal Investment Timing, Stochastic Models, Real Options.
MSC2010: 60G40, 62L15.

1 Introduction

The valuation of real options is a critical aspect of investment decision-making, particularly in environments characterized by significant uncertainties. Traditional approaches to investment valuation often rely on deterministic models that may inadequately capture real-world market risks and volatilities. In contrast, real options theory provides a more sophisticated framework that incorporates managerial flexibility in decision-making under uncertainty, paralleling the decision-making flexibility seen in financial options ([1–3]). A key source of uncertainty impacting investment decisions is currency uncertainty, also known as exchange rate risk, which arises from fluctuations in the exchange rates between two currencies. Such fluctuations can significantly impact the cash flows and profitability of investments, especially for multinational corporations and investors in foreign markets ([4]). Effective management of this risk is crucial for optimizing investment timing and valuation.

In this context, stochastic models play a crucial role in capturing the dynamic nature of of these economic variables ([5], [6]). These models provide the mathematical foundation for developing more robust investment valuation models that can account for the complexities associated with currency risk.

Optimal investment problems are central in the fields of mathematical finance and economics, focusing on identifying strategies for optimal investment timing. The literature on optimal investment problems is extensive, ranging from classical theories of risk-return trade-offs to more sophisticated models that incorporate complexities such as stochastic volatility, transaction costs, and strategic factors. Building on foundational works such as those by [7–9], research has increasingly examined the benefits of delaying investment decisions under uncertainty—a strategy made feasible through real options theory ([10–13], and [3]). This approach contrasts with traditional Net Present Value (NPV) methods, which often disregard the value of waiting in uncertain conditions. McDonald and Siegel's model ([7]) utilizes stochastic processes to describe investment opportunities, suggesting that the optimal timing of an investment is when the marginal benefit of waiting equals the marginal cost. This foundational idea has inspired numerous studies across diverse domains, including natural resources, technology, and finance.

Dixit and Pindyck ([1]) further advanced real options theory by examining how firms can leverage the option to delay investments until market conditions become favorable. However, their model assumes a relatively constant or minimal currency impact, which limits its application in multi-currency or volatile currency environments. In addressing this gap, subsequent studies have incorporated currency fluctuations into investment models to evaluate the impact of foreign exchange risk on optimal investment timing ([14–17]). As markets continue to evolve, the methodologies and models employed to solve optimal investment problems, with ongoing research focus on robustness, computational efficiency, and the integration of non-traditional factors into investment decisions. Research on managing currency risk includes studies that examine systematic risks associated with foreign currency-denominated returns, emphasizing the role of exchange rate fluctuations in complicating international asset evaluations. Another study addresses investment strategies in contexts where investors are ambiguity-averse and face currency uncertainty, demonstrating the benefits of robust optimization in such cases ([18]).

Similarly, recent research provides an overview of real options theory applications in uncertain international investments, advocating further research on investment decisions under fluctuating foreign exchange rates ([17]). Additionally, another study uses a real options model to show that higher variance and correlation between currency and project value volatility increase option values, encouraging firms to defer investments.

Studies also explore the effect of uncertainty on real option valuations, showing that increased uncertainty often leads firms to be more cautious in investment decisions. The empirical and numerical findings underscore the significance of currency fluctuations in influencing investment timing and valuation, especially for international projects. Furthermore, some research proposes a new estimator for quantifying the influence of currency volatility on investment decisions, adding a tool to the methodological approaches in this field ([16]).

One of the most critical yet underexplored aspects of investment uncertainty is currency risk, which arises due to fluctuations in exchange rates affecting firms engaged in international trade, foreign direct investment, or cross-border mergers and acquisitions. Industries such as energy, where companies invest in infrastructure projects across multiple currencies, and manufacturing, where firms rely on imported raw materials, are particularly exposed to exchange rate fluctuations. Despite the extensive literature on optimal investment timing, relatively few studies explicitly account for the dynamic interplay between exchange rate uncertainty and investment costs.

This paper addresses this gap by introducing a currency uncertainty ratio, which captures the impact of currency fluctuations on investment payoffs. We develop a stochastic optimal stopping model to analyze the timing of investment under two scenarios: (i) fixed investment costs and (ii) stochastic investment costs evolving with the project value. Our key contribution is the derivation of optimal investment thresholds and option values under currency uncertainty. We show that higher currency risk leads to more conservative investment strategies, delaying investment in uncertain markets. Our findings provide valuable insights for multinational firms, policymakers, and financial analysts involved in global investment planning.

This is especially relevant for firms operating in volatile currency markets, where the future value of cash flows may be highly sensitive to exchange rate changes, affecting profitability and competitiveness.

This paper is organized as follows: Section 2 presents notations and introduces the currency uncertainty ratio. Section 3 derives the proposed models. Section 4 provides sensitivity analysis, and Section 5 concludes.

2 Notations and Assumptions

In this section, we introduce the notations and assumptions that will be used throughout the paper, including the currency uncertainty ratio. These definitions and concepts provide a foundation for the optimal stopping models formulated in Section 3. We assume a continuous-time setting and a frictionless market.

Throughout, it is assumed that the stochastic processes are defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, where $\{\mathcal{F}_t\}$ is the filtration satisfying the usual conditions:

a) *Completeness*: every $\mathbb{P}$-null set in $\mathcal{F}$ belongs to $\mathcal{F}_0$ and thus to each $\mathcal{F}_t$.

b) *Right-continuity*: $\mathcal{F}_t = \mathcal{F}_{t^+} \equiv \bigcap_{s>t} \mathcal{F}_s \quad \forall t \geq 0$.

Next, we introduce definitions and theorems relevant to optimal stopping problems in a stochastic environment.

Definition 2.1. *The process X_t is said to follow a geometric Brownian motion if it satisfies the stochastic differential equation (SDE):*

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t,$$

where μ is the drift parameter, σ is the volatility parameter, and W_t is a standard Wiener process under the probability measure $\mathbb{P}$. This process will represent either the project value or the currency ratio in subsequent sections.

Lemma 2.1 (Ito's Lemma)**.** *Given a twice-differentiable function $f(t, X_t)$ and an SDE $dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dW_t$, Ito's Lemma states that the differential of f is:*

$$df(t, X_t) = \frac{\partial f}{\partial t}\, dt + \frac{\partial f}{\partial x}\, dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma(X_t)^2\, dt.$$

This lemma is essential for deriving the dynamics of functions of stochastic processes within our model.

Theorem 2.2 (Infinitesimal Generator of a Diffusion Process)**.** *The infinitesimal generator $\mathcal{L}$ of a diffusion process X_t governed by the SDE $dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dW_t$ is defined by:*

$$\mathcal{L}f(x) = \mu(x)\frac{df}{dx} + \frac{1}{2}\sigma(x)^2\frac{d^2 f}{dx^2},$$

for a sufficiently smooth function f. This operator plays a key role in optimal stopping formulation.

Definition 2.3 (Stopping Time)**.** *A non-negative random variable $\tau : \Omega \to \mathbb{R}_+ \cup \{\infty\}$ is called a stopping time if:*

$$\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t, \quad \forall\, t \geq 0.$$

Hitting times are examples of stopping times. The hitting time of level x by the process $\{X_t\}_{t\in\mathbb{R}_+}$, defined as

$$\tau_x := \inf\{t \geq 0 : X_t = x\},$$

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is a stopping time. We will often work with first hitting times, defined by

$$\tau_x := \inf\{t > 0 : X_t > x\},$$

which gives the first time a process attains a value greater than x.

Definition 2.4 (Optimal Stopping Time)**.** *An optimal stopping time τ^* is the time at which it is optimal to stop a stochastic process X_t to maximize the expected reward. Mathematically, the optimal stopping problem is formulated as finding τ^* that maximizes the value function:*

$$\mathcal{V}(x) = \sup_{\tau} \mathbb{E}\left[e^{-r\tau} g(X_\tau) \mid X_0 = x\right],$$

where $g(X_\tau)$ is the payoff function, and r is the discount rate.

2.1 General Optimal Stopping Problem Formulation

Optimal stopping problems consist of finding the best possible payoff and the time at which this payoff can be achieved. The goal is to determine a stopping time τ, defined with respect to $\{\mathcal{F}_t\}$, that maximizes the expected reward function:

$$\mathcal{V}(x) = \sup_{\tau} \mathbb{E}\left[e^{-r\tau} g(X_\tau) \mid X_0 = x\right],$$

where τ is the stopping time that maximizes the expected reward; $\mathcal{V}(x)$ represents the value function, or maximum expected reward, given the initial state $X_0 = x$; $g(X_\tau)$ is the payoff function at the stopping time τ; and $e^{-r\tau}$ is the discount factor, with $r \geq 0$ representing the discount rate. The value function $\mathcal{V}(x)$ satisfies a Hamilton-Jacobi-Bellman (HJB) equation. For a diffusion process X_t with infinitesimal generator $\mathcal{L}$, the HJB equation for $\mathcal{V}(x)$ is:

$$\mathcal{L}\mathcal{V}(x) + r\mathcal{V}(x) = 0, \quad x \in \text{Continuation Region},$$

where the continuation region is the set of states where stopping is not optimal. The boundary conditions are defined on a free boundary, which represents the set of states where it becomes optimal to stop.

Solving the free-boundary problem involves identifying the stopping region (where stopping is optimal) and the continuation region (where the process should continue).

Example 2.2. Consider the arbitrage-free price of the perpetual American call option that marches the optimal stopping problem

$$\mathbb{V}(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left(e^{-qa} g\left(B_\tau\right)\right) \tag{2.1}$$

where a is a stopping time and B a geometric started from $B_0 = x$ solving

$$dB_t = qB_t dt + \sigma B_t dW_t.$$

Furthermore, g is the nonnegative continuous reward function (payoff function) defined by $g(x) = (x - K)^+$.

To ensure existence of the expectation in (2.1) the following assumption is made:

$$\mathbb{E}\left(\sup_{t \leq \tau \leq T} |g_\tau|\right) < \infty \tag{2.2}$$

(with $g_T \equiv 0$ when $T = \infty$).

The optimal stopping time of problem (2.1) can also be represented as the first time of the process B_t leaves the continuation set C (or the first time B enters D the stopping set). The optimal time to stop is:

$$\tau_D = \inf\{t \geq 0 : B_t \in D\} = \inf\{t \geq 0 : B_t \notin C\}. \tag{2.3}$$

It follows from the literature (see [19–23]) that the optimal stopping time (whenever it exists) is given by

$$\tau_b = \inf\{t \geq 0 : X_t \geq b\} \tag{2.4}$$

where b is the free boundary which is not known. Then $\mathbb{V}$ and b satisfy the following boundary conditions:

$$\mathcal{A}_B \mathbb{V} = q\mathbb{V} \qquad x < b \tag{2.5}$$

$$\mathbb{V}(x) = (x-K)^+ \qquad x = b \tag{2.6}$$

$$\mathbb{V}'(x) = g'(x) \qquad x = b\,(smooth\,fit) \tag{2.7}$$

$$\mathbb{V}(x) > (x-K)^+ \qquad x < b \tag{2.8}$$

$$\mathbb{V}(x) = (x-K)^+ \qquad x \geq b, \tag{2.9}$$

where the infinitesimal operator $\mathcal{A}$ of the process $B$is given by

$$\mathcal{A}_B = qx\frac{\partial}{\partial x} + \frac{\sigma^2}{2}x^2\frac{\partial^2}{\partial x^2}.$$

Condition (2.5) is called the asset equilibrium condition. (2.6) is the value matching condition. (2.7) is the smooth pasting condition. (2.8) the continuation region (value function dominates the gain function) and (2.9) the value matching condition (Instantaneous stopping beyond the boundary). Here it is assumed that both V and g are continuous and smooth.

2.2 Currency Uncertainty Ratio

Consider a situation where you are receiving future cash flows in your home (domestic) currency (say Nigerian Naira, NG) but the value of these cash flows is affected by fluctuations in a foreign currency (say US dollar, USD). This implies that you are receiving future cash flows in naira, but the exchange rate between naira and USD affects the value of these cash flows. To account for the impact of foreign currency fluctuations on your home currency cash flows, we have:

$$CF_{home,t} = CF_{home,t} \times \left(\frac{E_{t0}}{E_t}\right), \tag{2.10}$$

where: $CF_{home,t}$ is the future cash flow in naira at time t, E_{t0} is the initial exchange rate (naira per USD), and E_t is the exchange rate at time t (naira per USD). This expression (2.10) shows that the value of future cash flows in your home currency depends on both the amount of cash flow in the foreign currency and the exchange rate at that time.
Example: Suppose initially, (E_{t0}): 500 naira = 1 USD and the future cash flow is ($CF_{home,t}$) = 100,000 naira. After a short time interval, currency fluctuation in the market will result to a new exchange rate of, say, (E_t): 1500 naira = 1 USD. With this new exchange rate, the adjusted future cash flow in naira would be:

$$CF_{home,t} = 100,000 \times \left(\frac{500}{1500}\right) = 100,000 \times \frac{1}{3} = 33,333.33 \text{ naira}$$

In this case, the naira has depreciated against the USD from 500 naira per USD to 1500 naira per USD. As a result, the value of the future cash flows in naira, when adjusted for the exchange rate change, decreases from 100,000 naira to approximately 33,333.33 naira. This demonstrates how the depreciation of the home currency (naira) against the foreign currency (USD) reduces the real value of your future cash flows in your home currency.

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Remark 2.5. *Exchange rates represent the value of one currency in terms of another, and they are always positive numbers. Thus, the ratio* $\frac{E_{t0}}{E_t}$ *will always be greater than 0 for all* t*. In consequent sections, we shall denote* $D_t = \frac{E_{t0}}{E_t}$.

We shall assume the following specific form for the project value ([4]):

$$V = D_t C_f.$$

In Section 3, we use this ratio to determine the optimal investment strategy under currency uncertainty.

3 Formulation of Models with Currency Uncertainty

In this section we give a general formulation of the dynamics of the future cash flows (called project value hereafter) and the investment cost. We shall consider different dynamics with currency uncertainty.

3.1 Case 1: V with GBM Dynamics and I is consant

We start by considering an investment problem where the project value, V, follows a Geometric Brownian Motion (GBM) and the investment cost, I is constant.
The model for the project value is given by

$$dV_t = r_v V_t dt + \sigma_v V_t dW_v(t), \qquad V(0) = v \tag{3.1}$$

Here we assume that the model is under an appropriate risk neutral measure with r_v, a risk neutral drift, σ_v is the volatility and $W_v(t)$ is a standard Wiener process (Brownian motion).
The solution to the SDE (3.1), is given by

$$V_t = v \exp\left(\left(r_v - \frac{\sigma_v^2}{2}\right) t + \sigma_v W(t)\right) \tag{3.2}$$

The payoff, which is also referred to as the performance criterion is given below.

Lemma 3.1. *The expected discounted net payoff is given by*

$$\mathcal{P}_1(V) = \mathbb{E}\left[e^{-r\tau} \mathcal{G}_1\left(V(\tau), I\right)\right] \tag{3.3}$$

where

$$\mathcal{G}_1\left(v\right) = \frac{vD}{r - r_v} - I, \qquad r - r_v > 0 \tag{3.4}$$

Proof. First we have that $E\left[V_t\right] = v \exp\left(r_v t\right)$ and $\mathbb{E}\left[e^{-rt} V_t\right] = v$ under a risk neutral measure. Thus,

$$\begin{aligned}
\mathcal{G}_1(v) &= \mathbb{E}\left[\left\{\int_0^\infty e^{-rt} D V_{t+\tau} dt\right\} - I\right]. \\
&= \mathbb{E}\left[e^{-r\tau} \mathbb{E}\left(\int_0^\infty e^{-rt}\left(D V_{t+\tau} | V_\tau\right) dt\right)\right] - I \\
&= \mathbb{E}\left[e^{-r\tau} \int_0^\infty e^{-rt} D\, \mathbb{E}\left(V_{t+\tau} | V_\tau dt\right)\right] - I \\
&= \mathbb{E}\left[e^{-r\tau} \frac{D\, V_\tau}{r - r_v}\right] - I \\
&= \frac{vD}{r - r_v} - I
\end{aligned} \tag{3.5}$$

This is done by applying Fubini's theorem, the tower property of conditional expectation and strong Markov property. □

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The optimal stopping problem is to find the value function, F(V,I) and an optimal stopping time, τ^* such that

$$F_1(V) = \sup_{\tau \in \mathcal{T}} \mathcal{P}_1^{\tau}(v) = \mathcal{P}_1^{\tau^*}(v) \tag{3.6}$$

The infinitesimal generator $\mathcal{A}_1$ of $V(v)$ is given by:

$$\mathcal{A}_1 f(v) = r_v v f'(v) + \frac{1}{2}\sigma_v^2 v^2 f''(v)$$

Given the form of the PDE: $r_v v f'(v) + \frac{1}{2}\sigma^2 v^2 f''(v) = 0$, we try a function of the form $f(v) = Av^k$, for $k > 0$ and some constant A. We get

$$\begin{aligned} \mathcal{A}_1 f(v) &= r_v A v(k v^{k-1}) + \frac{1}{2}\sigma_v^2 A v^2 (k(k-1)v^{k-2}) \\ &= Av^k \left(r_v k + \frac{1}{2}\sigma_v^2 k(k-1) \right) \\ &= Av^k h(k) \end{aligned} \tag{3.7}$$

Following [21]), in the continuation region, for $\rho > 0$,

$$\mathcal{A}_1 f(v) = \rho f(v).$$

Since, $\mathcal{A}_1 f(v) = Av^k h(k) = f(v)h(k)$, then $h(k) - \rho = 0$ which can be written as

$$\frac{1}{2}\sigma_v^2 k(k-1) + r_v k - \rho = 0.$$

The roots are

$$k_1 = \frac{1}{2} - \frac{r_v}{\sigma_v^2} + \sqrt{\left(\frac{r_v}{\sigma_v^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma_v^2}} > 0 \qquad k_2 = \frac{1}{2} - \frac{r_v}{\sigma_v^2} - \sqrt{\left(\frac{r_v}{\sigma_v^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma_v^2}} < 0.$$

Then since $k_2 < 0$, $f(v) = Av^{k_1}$.
The following proposition gives an explicit expression for the value of the investment under the dynamics described with currency uncertainty.

Proposition 3.1. *The value of the investment option is given by*

$$F_1(v) = \begin{cases} \dfrac{vD}{r - r_v} - I & \text{if} \quad v \geq v^* \\ Av^{k_1} & \text{if} \quad v < v^* \end{cases} \tag{3.8}$$

where $k_1 = \frac{1}{2} - \frac{r_v}{\sigma_v^2} + \sqrt{\left(\frac{r_v}{\sigma_v^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma_v^2}}$ *is the positive root of the equation* $h(k) = \rho$, *and*

$$A = \frac{\phi_1^k\left((k-1)\right)^{k-1}}{I^{k-1}k^k} > 0, \tag{3.9}$$

$$v^* = \frac{Ik}{\phi_1(k-1)} > 0, \tag{3.10}$$

for $\phi_1 = \frac{D}{r - r_v}$.
Additionally, the optimal stopping time $\tau^* = \inf\{t \geq 0 : V_t \geq v^*\}$.

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Proof. Applying the following boundary conditions and computing the derivatives we get:

$$F_1(v^*) = \frac{v^*D}{r - r_v} - I, \qquad F_1'(v^*) = \frac{D}{r - r_v} = \phi_1,$$

equating with

$$F_1(v^*) = Av^{*k} - I, \qquad F_1'(v^*) = Akv^{*k-1},$$

then simplify and the results follows. □

Note that when $\phi_1 = 1$, the result is the same as that of ([1]).

3.2 Case 2: V and I with GBM Dynamics

This section presents the formulation and solution of the optimal investment problem where the investment project value and investment cost evolve stochastically. We derive the value function, reduce the problem to one dimension, solve the associated Hamilton-Jacobi-Bellman (HJB) equation, and propose a closed-form solution for the optimal stopping problem.

The dynamics of the project value, V_t and investment cost, I_t are modeled as GBMs with no correlation. We have the risk neutral models as follows:

$$dV_t = r_v V_t dt + \sigma_v V_t dW_t^v$$

and

$$dI_t = r_I I_t dt + \sigma_I I_t dW_t^I$$

where $r_v, r_I, \sigma_v, \sigma_I$, are the drift and volatility terms and $W_v(t), W_I(t)$ are independent standard Brownian motions under the risk-neutral probability measure.

Lemma 3.2. *The expected discounted net payoff is given by*

$$\mathcal{P}_2(V, I) = \mathbb{E}\left[e^{-r\tau}\mathcal{G}_2\left(V(\tau), I(\tau)\right)\right] \tag{3.11}$$

where

$$\mathcal{G}_2\left(v, i\right) = \frac{vD}{r - r_v} - \frac{i}{r - r_I} \tag{3.12}$$

Proof. The proof follows the same idea as the proof of Lemma 3.1 Given the dynamics of V_t and I_t, the aim is to maximize the expected discounted value:

$$J(v_0, i_0) = \mathbb{E}_{(v_0,i_0)}\left[\int_\tau^\infty e^{-rt}\left(V_tD - I_t\right)\, dt\right],$$

where D is a currency ratio, and r is the discount rate. Note that

$$V_t = v_0 e^{\mu_v t + \sigma_v W_t^v} \Rightarrow \mathbb{E}[v_t] = v_0 e^{\mu_v t}.$$

and

$$I_t = i_0 e^{\mu_I t + \sigma_I B_t^I} \Rightarrow \mathbb{E}[I_t] = i_0 e^{\mu_I t}.$$

Thus, we have

$$\begin{aligned}
\mathcal{G}_2\left(v, i\right) &= \mathbb{E}_{(v_0,i_0)}\left[\int_\tau^\infty e^{-rt}\left(V_tD - I_t\right)\, dt\right] \\
&= \mathbb{E}\left[\int_\tau^\infty e^{-rt}V_tD\, dt\right] + \mathbb{E}\left[\int_\tau^\infty e^{-rt}I_t\, dt\right] \\
&= Dv_0\int_\tau^\infty e^{-(r-\mu_v)t}\, dt + i_0\int_\tau^\infty e^{-(r-\mu_I)t}\, dt \\
&= \frac{Dv_0}{r - \mu_v} + \frac{i_0}{r - \mu_I}
\end{aligned} \tag{3.13}$$

□

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The optimal stopping problem is to find the value function $F_2(V, I)$ and an optimal stopping time τ^* such that:

$$F(V, I) = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-r\tau}\mathcal{G}_2(V_\tau, I_\tau)\right], \tag{3.14}$$

where the payoff function is:

$$\mathcal{G}_2(V, I) = \frac{VD}{r - \mu_V} - \frac{I}{r - \mu_I}.$$

The Generator $\mathcal{A}_2$ for the joint process (V_t, I_t) is given by:

$$\mathcal{A}_2 F(V, I) == \mu_V V \frac{\partial F}{\partial V} + \mu_I I \frac{\partial F}{\partial I} + \frac{1}{2}\sigma_V^2 V^2 \frac{\partial^2 F}{\partial V^2} + \frac{1}{2}\sigma_I^2 I^2 \frac{\partial^2 F}{\partial I^2}.$$

In the continuation region, the value function $F(V, I)$ satisfies the Hamilton-Jacobi-Bellman (HJB) equation:

$$rF(V, I) = \mathcal{A}_2 F(V, I),$$

which explicitly becomes:

$$rF(V, I) = \mu_V V \frac{\partial F}{\partial V} + \mu_I I \frac{\partial F}{\partial I} + \frac{1}{2}\sigma_V^2 V^2 \frac{\partial^2 F}{\partial V^2} + \frac{1}{2}\sigma_I^2 I^2 \frac{\partial^2 F}{\partial I^2}.$$

To simplify the problem we can write $\mathcal{G}_2(v, i) = i\left(\frac{vD}{i(r-\mu_V)} - \frac{1}{r-\mu_I}\right) = i\left(\frac{yD}{(r-\mu_V)} - \frac{1}{r-\mu_I}\right) = ig_2(y)$ and where $y = \frac{v}{i}$ and assume:

$$F(v, i) = if(y).$$

Differentiating and substituting into the HJB equation, the problem reduces to:

$$\frac{1}{2}\sigma_y^2 y^2 f''(y) + \mu_y y f'(y) - (r - \mu_I)f(y) = 0, \tag{3.15}$$

where: $\sigma_y^2 = \sigma_V^2 + \sigma_I^2, \quad \mu_y = \mu_V - \mu_I$.

3.2.1 Solution to the HJB Equation

The reduced HJB equation (3.15) in one dimension is a Cauchy-Euler equation. The general solution in the continuation region is:

$$f(y) = A_1 y^{\beta_1} + A_2 y^{\beta_2},$$

where $\beta_1 > 0$ and $\beta_2 < 0$ are the roots of the characteristic equation and A_1, A_2 are constants to be determined from boundary conditions. To ensure economic feasibility, we require $A_2 = 0$, as the negative root β_2 leads to unbounded behavior as $y \to 0^+$. So we assume a solution of the form:

$$f(y) = A_1 y^\beta,$$

where β is to be determined. The characteristic equation is given by

$$\frac{1}{2}\sigma_y^2\beta^2 + \left(\mu_y - \frac{1}{2}\sigma_y^2\right)\beta - (r - \mu_I) = 0.$$

The roots of the equation are:

$$\beta = \frac{-\left(\mu_y - \frac{1}{2}\sigma_y^2\right) \pm \sqrt{\left(\mu_y - \frac{1}{2}\sigma_y^2\right)^2 - 2\sigma_y^2(r - \mu_I)}}{\sigma_y^2}.$$

We denote the two roots the positive and negative roots as β_1 and β_2 respectively.

We now present the main resukt for this section as follows.

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Proposition 3.2. *The value function of the investment $F(V, I)$ is given by:*

$$F(V,I) = \begin{cases} A_1 V^{\beta_1} I^{1-\beta_1}, & \textit{if } \frac{V}{I} < y^*, \\ \frac{VD}{r-\mu_V} - \frac{I}{r-\mu_I}, & \textit{if } \frac{V}{I} \geq y^*, \end{cases}$$

where:

$$y^* = \frac{\beta_1}{(r-\mu_I)} \cdot \frac{(r-\mu_V)}{D(\beta_1 - 1)},$$

$$A_1 = \frac{1}{(r-\mu_I)(\beta_1-1)} \cdot \left(\frac{D(\beta_1-1)}{\beta_1(r-\mu_V)}\right)^{\beta_1},$$

$$\beta_1 = \frac{-\left(\mu_y - \frac{1}{2}\sigma_y^2\right) + \sqrt{\left(\mu_y - \frac{1}{2}\sigma_y^2\right)^2 - 2\sigma_y^2(r-\mu_I)}}{\sigma_y^2},$$

and:

$$\mu_y = \mu_V - \mu_I, \quad \sigma_y^2 = \sigma_V^2 + \sigma_I^2.$$

The value function $F(V, I)$ provides the optimal investment strategy under currency uncertainty. The stopping threshold y^* and the coefficient A_1 capture the effects of volatility, drift, and currency ratio D. This result extend classical optimal stopping models by incorporating stochastic investment costs and exchange rate dynamics.

Proof. To derive the value function, we start with the reduced HJB equation (3.15) to get the roots as seen above. Next, in the stopping region ($y \geq y^*$), the value function equals the payoff:

$$F(V,I) = \frac{VD}{r-\mu_V} - \frac{I}{r-\mu_I}.$$

Thus, to determine A_1 and y^*, we use the boundary conditions:
Value Matching at $y = y^*$

$$A_1(y^*)^{\beta_1} = \frac{y^* D}{r-\mu_V} - \frac{1}{r-\mu_I}.$$

and Smooth Pasting at $y = y^*$

$$\beta_1 A_1 (y^*)^{\beta_1 - 1} = \frac{D}{r-\mu_V}.$$

Solving these equations, we find:

$$y^* = \frac{\beta_1}{(r-\mu_I)} \cdot \frac{(r-\mu_V)}{D(\beta_1-1)},$$

and:

$$A_1 = \frac{1}{(r-\mu_I)(\beta_1-1)} \cdot \left(\frac{D(\beta_1-1)}{\beta_1(r-\mu_V)}\right)^{\beta_1}.$$

□

4 Numerical Analysis

In this section, we present a numerical analysis of the optimal stopping problem under currency uncertainty. For our analysis, we assume a set of parameters for the project value and investment cost dynamics, as well as the currency uncertainty ratio. For the computation of parameters, see [22, 23]).

4.1 Case 1

Figure 1 shows the investment opportunity value $F_1(v)$ as a function of the project value v for a fixed currency ratio $D = 0.2$. $F_1(v)$ follows two distinct forms depending on whether v is below or above the threshold v^*: For $v < v^*$, the value $F_1(v)$ follows a power function AV^{k_1}, indicating that the investment value is low and the project may not yet be viable. For $v \geq v^*$, $F_1(v)$ increases linearly, reflecting a more favorable condition where the project value supports an immediate investment decision. This figure demonstrates that, for a fixed D, the project must reach a certain value v^*, critical value, to be considered for investment.

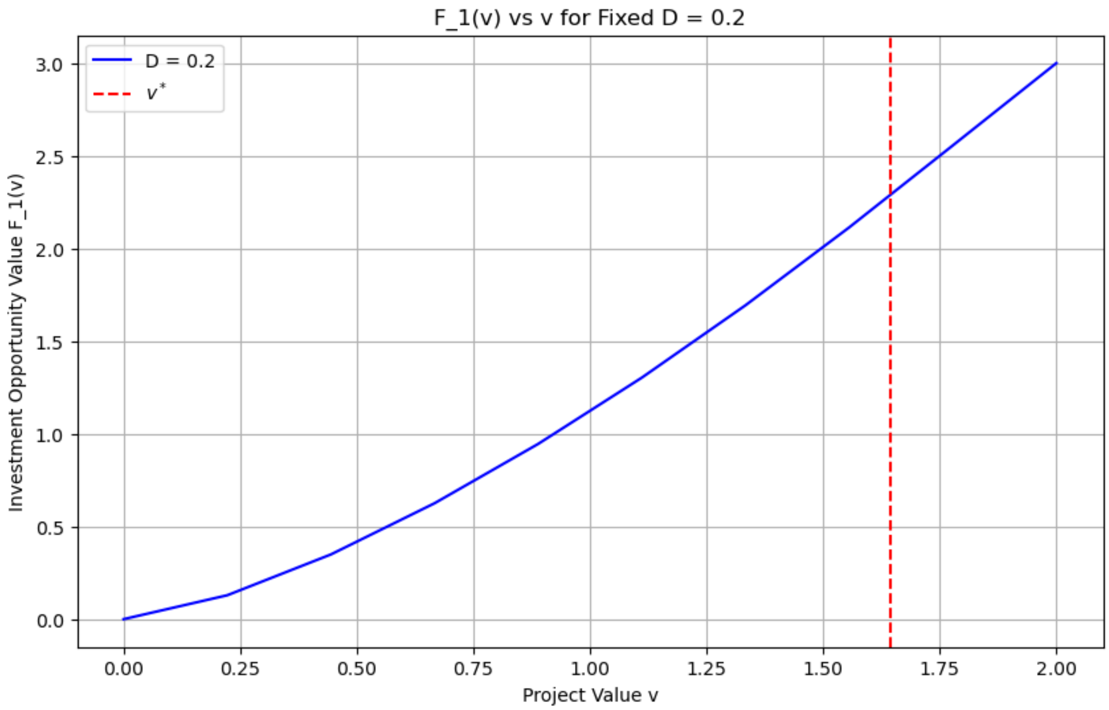

Figure 1: $F_1(v)$ vs. v

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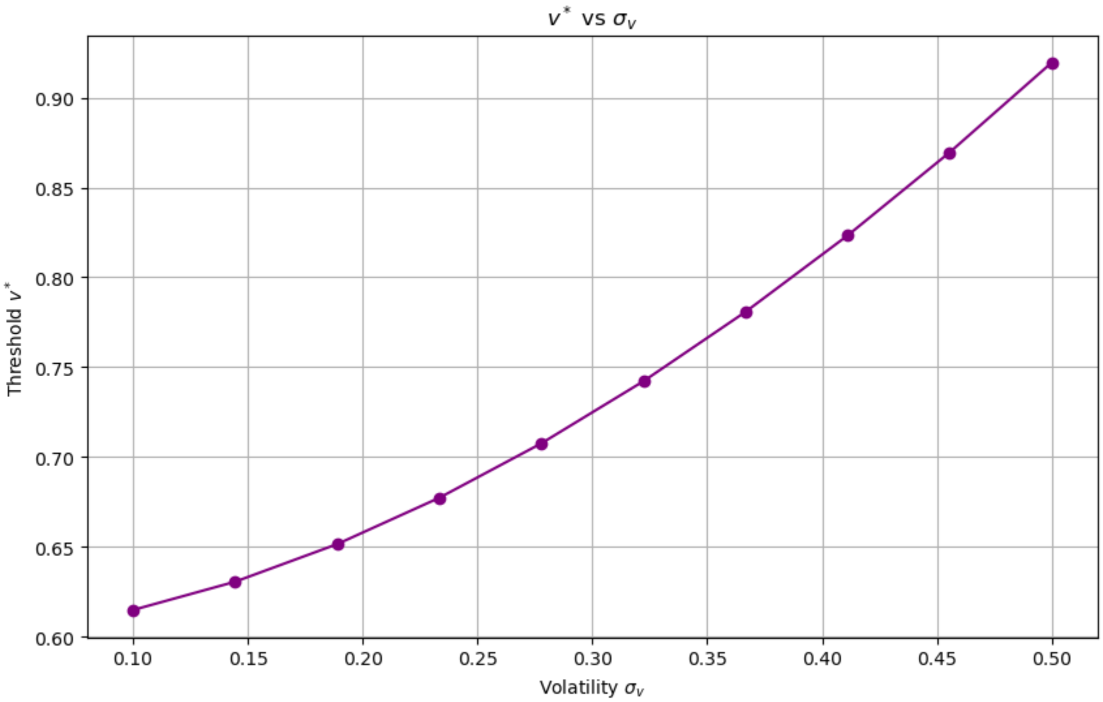

Figure 2: The threshold v^* as a function of volatility σ_v

Figure 2 shows the threshold v^*, the minimum project value required to trigger investment, as a function of volatility σ_v. The threshold v^* increases as σ_v rises. This implies that with greater volatility, the project value must reach a higher level before the investment becomes optimal. This increase in v^* reflects greater caution by investors, who may delay investment until the project's value justifies the heightened risk due to volatility. The relationship between v^* and σ_v is also well-documented in the real options literature. Higher volatility typically increases the 'hurdle rate', or investment threshold, as investors weigh the benefits of waiting against potential risks ([7,8]). This strategic delay is characteristic of investment in volatile environments.

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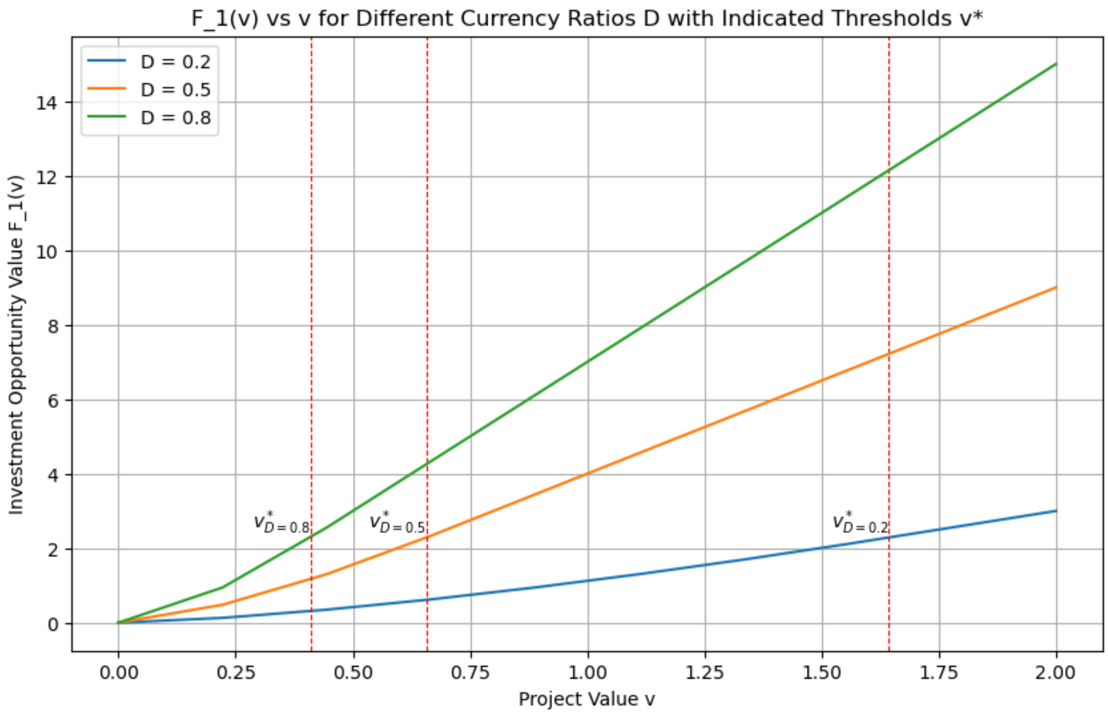

Figure 3: $F_1(v)$ with D = 02, 0.5, 0.8

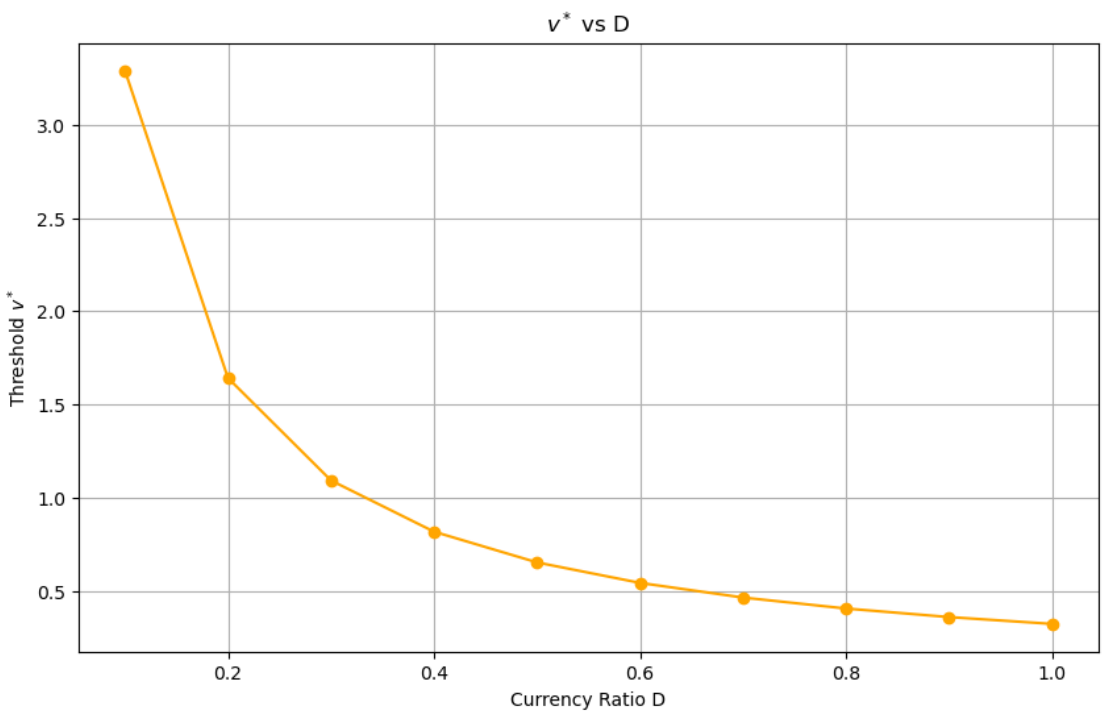

Figure 4: Effect of D on the threshold v^*

The currency ratio, D, which reflects the influence of foreign exchange fluctuations on the

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value of future cash flows in the domestic currency, which subsequently affects the timing and viability of investment decisions. Figure 3 shows the $F_1(v)$ against v for varying currency ratios $D = 0.2, 0.5, 0.8$. Observing the curves: Higher D values lead to higher $F_1(v)$ values and a lower threshold v^*. This implies that when the currency ratio is favorable (strong home currency), investment is viable at lower project values. Lower D values result in lower $F_1(v)$ values and higher thresholds v^*, reflecting a cautious approach to investment in scenarios where the home currency is weaker. This figure illustrates how increasing D (indicating a favorable currency environment) makes the investment opportunity more attractive and reduces the required project value v needed to trigger investment.

This finding is consistent with the real options literature, which emphasizes that currency strength can enhance the expected payoff of investments ([15, 16]). When exchange rates favor the home currency, investment thresholds can lower, and the value of waiting for investment becomes reduced.

In Figure 4, we analyze the relationship between the currency ratio D and the investment threshold v^*. As D increases, v^* decreases, meaning the project value threshold for investment is lower when the home currency is strong. We notice here that a higher currency ratio (stronger home currency) reduces the required project value for investment, signaling that investments become feasible at lower project values when exchange rates favor the home currency. Conversely, when D is low (weaker home currency), v^* increases, requiring the project value v to be higher before investment is optimal. This effect illustrates that a favorable currency ratio D encourages earlier investment, as future cash flows in the home currency hold more value.

This behavior supports the findings in international finance, particularly in studies involving exchange rate risk and investment thresholds. When the home currency appreciates, the real value of expected cash flows in the home currency increases, effectively lowering the project value needed to justify investment ([15, 17]).

The problem is to determine the point at which it is optimal to invinvest in return for an asset worth V. Since v evolvestochastically. we will not be able to determine time T as we did above. Instead. our investment rule will take the form of In this section, we provide some numerical results and analyze the impact of some financial model parameters on the robust optimal investment strategies. For convenience, we consider the value of the robust optimal investment strategies at time t = 0. In general, we can also suppose that t is a positive constant. In this case, we can obtain the numerical results of the optimal investment strategies by using the same method. The greater the volatility, the greater the risk of stock price. Hence, the investors will naturally reduce their investment in domestic stocks when it increases.

4.2 Case 2

The analysis presented here are based on the model where both the project value (V) and investment cost (I) follow GBMs. We compare these findings with existing literature and highlight the differences between this case and the scenario where I is constant (case 1).

Figure 5 shows the behavior of $F(V, I)$ as V varies for fixed I. The value function increases as V grows within the continuation region, reflecting the increasing project value. The dashed line represents the threshold y^*I, beyond which it becomes optimal to stop and invest. This result is consistent with Dixit and Pindyck (1994), where the value function in optimal stopping problems exhibits monotonicity before the stopping boundary. Figure 6 illustrates the effect of varying D (currency ratio) on $F(V, I)$. Higher values of D lead to lower value functions, reflecting reduced currency uncertainty. This is consistent with Grenadier and Malenko (2010), where stable currencies reduce the strategic value of waiting.

Figure 7 shows that the optimal investment threshold y^* increases with σ_V. This result is consistent with Dixit and Pindyck (1994), which highlights that higher volatility delays optimal investment decisions due to increased uncertainty.

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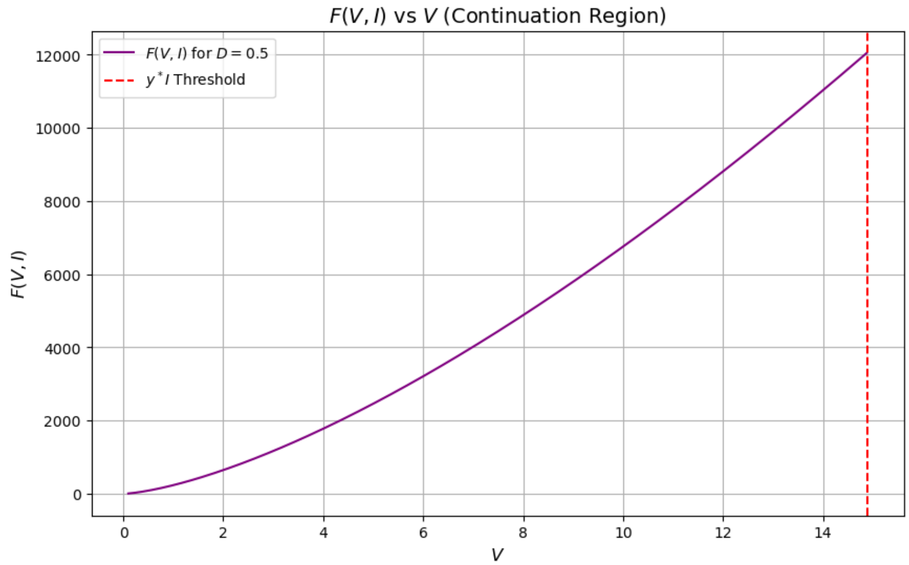

Figure 5: Plot of $F(V, I)$ vs V for fixed I in the continuation region. The dashed line indicates the investment threshold y^*I.

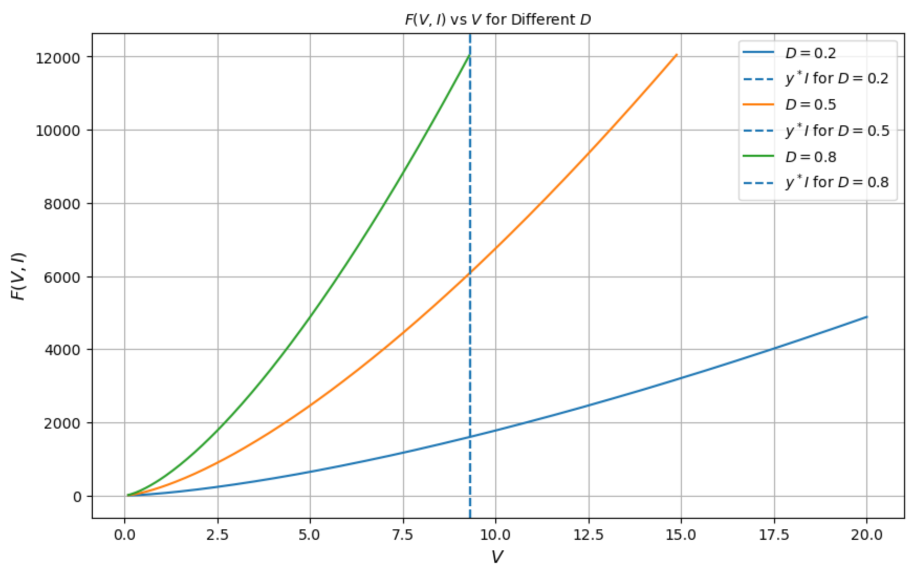

Figure 6: Plot of $F(V, I)$ vs V for different values of D in the continuation region.

Figure 8 demonstrates that y^* decreases as D increases. This result aligns with Grenadier and Malenko (2010), where stable exchange rates promote earlier investment.
In case 1, where I is constant, the dynamics are simpler, and the value function $F(V, I)$ depends solely on the behavior of V. In contrast, in case 2, both V and I follow GBMs, introducing an additional layer of uncertainty. The inclusion of I in the stochastic framework increases the flexibility of investment decisions, as evidenced by the sensitivity of $F(V, I)$ and y^* to parameters such as D and σ_V as shown in Figures 6 and 8. However, this complexity requires more computational effort, as highlighted in Miao and Wang (2007). Case 2 provides more realistic modeling for investment problems involving fluctuating costs, consistent with Chen and Wang (2019).

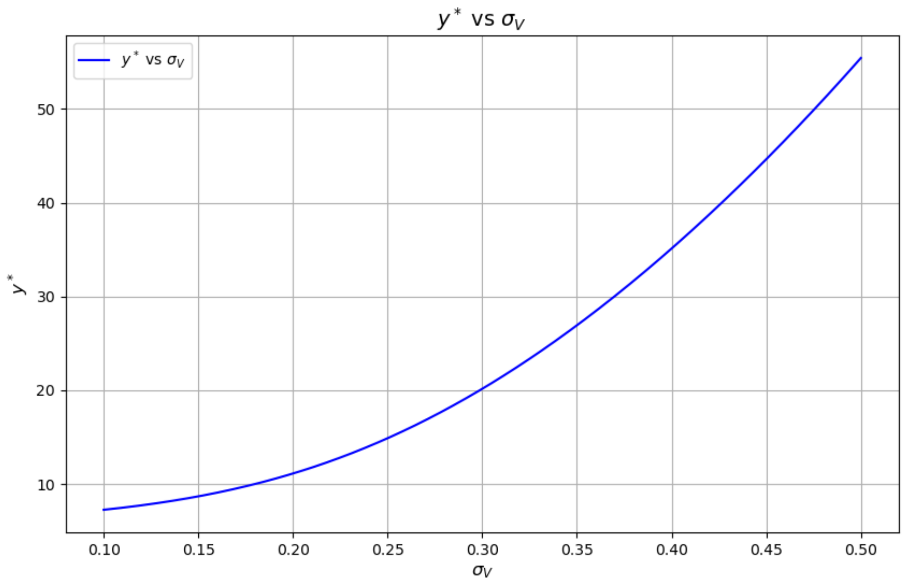

Figure 7: Plot of y^* vs σ_V (volatility of V).

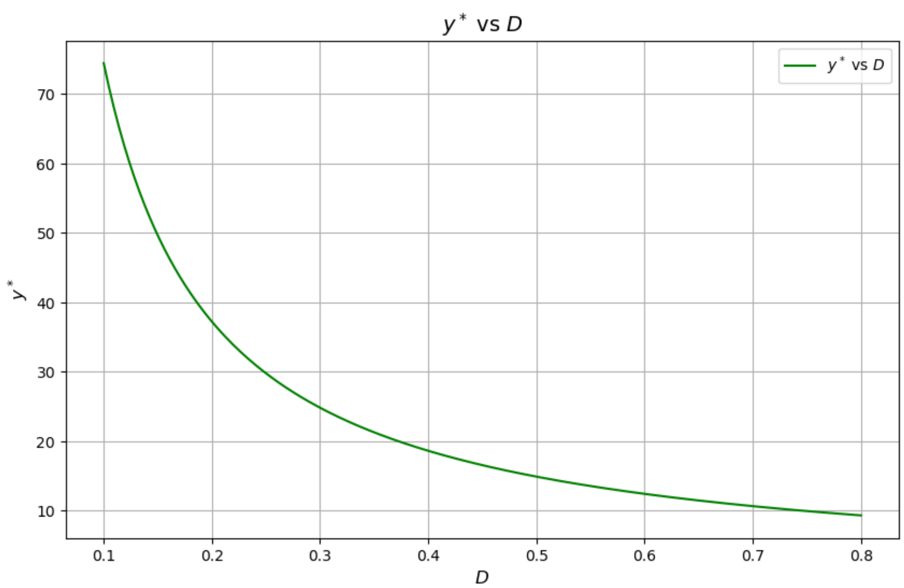

Figure 8: Plot of y^* vs D (currency ratio).

When investment cost is constant, investment timing primarily depends on project value fluctuations and currency risk, leading to delayed investment in high volatility environments. When both project value and investment cost follow a geometric Brownian motion (GBM), the decision-making process is more complex as both revenue and cost streams are stochastic, increasing uncertainty and risk exposure. The stochastic cost case shows that investment thresholds are generally higher than in the constant-cost case, indicating that firms are more cautious before committing resources in highly volatile environments.

The practical implications in real-world applications is that industries with relatively stable investment costs (e.g., infrastructure projects, manufacturing plants) are better modeled with the constant investment cost framework. Sectors where investment costs fluctuate significantly (e.g., energy, commodity markets, and construction) align more with the stochastic investment cost model, since both raw materials and labor costs are volatile over time. Model

One major assumption is the frictionless market, which may not reflect real-world conditions such as transaction costs, regulatory constraints, and liquidity issues. Our model does not incorporate jump processes or regime-switching behavior in currency fluctuations, which are known to affect investment timing significantly. Future research will extend the model by considering correlation between project value and investment cost, introducing jump processes, and integrating empirical calibration with historical data to improve applicability.

5 Conclusion

This paper provides an analysis of the optimal investment problem under currency uncertainty by examining two distinct cases: a model where the project value V evolves as a GBM while the investment cost I remains constant, and a model where both V and I evolve as GBMs. The two cases represent scenarios of differing complexity, offering insights into the interplay between investment value and currency fluctuations. The results underscore the importance of understanding the currency ratios on investment decisions. Incorporating currency uncertainty into investment models enhances their applicability in real-world financial settings, especially for multinational corporations and investors dealing with cross-border projects. Future research could extend this work to include jumps in project values and costs or explore optimal investment strategies under correlated stochastic processes for V and I. We acknowledge the model's assumption of continuous price movements. Future research will incorporate jump processes and correlated stochastic variables to enhance realism. Empirical validation and calibration with real-world financial data will also be considered in subsequent work.

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