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Some New Operations on Picture Fuzzy Multisets

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Abstract

A multiset is an extension of a crisp set in which elements are allowed to occur more than once, enabling the representation of data where frequency or multiplicity is essential. Building upon this, fuzzy multisets introduce a degree of membership to each element, capturing both quantity and uncertainty. Further developments have led to the formulation of picture fuzzy multisets a robust framework that extends picture fuzzy sets. In this paper, we propose some new operations on picture fuzzy multisets by analyzing existing operations on fuzzy sets, and fuzzy multisets, extending these foundations using picture fuzzy logic principles to define operations that maintain the consistency of positive membership, neutral membership, and negative membership and refusal degrees, establishing a set of axioms and properties (such as commutativity, and distributivity) that the operations must satisfy, proving theorems that verify these properties and constructing a detailed numerical example to illustrate and validate the behavior and correctness of the proposed operations. It was shown that the proposed operations are well-defined, internally consistent, and closed under the structure of PFMs. The example demonstrates that the operations handle ambiguity, contradiction, and repetition effectively, making them suitable for applications in multi-criteria decision-making, knowledge representation, and information systems. The new operations significantly broaden the mathematical toolkit available for handling picture fuzzy multisets. They lay a foundation for future research into more complex structures such as picture fuzzy multirelations, aggregation operators, and soft computing models based on picture fuzzy multisets.

Keywords: Multiset, Fuzzy multiset, Picture fuzzy Set, Fuzzy Set.
MSC2010: 03E72, 08A72.

1 Introduction

Zadeh, 1965 [1] introduced the theory of fuzzy sets as a generalisation of crisp sets. The theory only takes into consideration membership degree of an element belonging to a particular set. In [2], an extension of fuzzy set to fuzzy parameterized soft expert set was established with an application to decision making. In [3], fuzzy set was applied to differential equations to find solution of first order initial value problems. Atanassov [4], extended the work of Zadeh to the theory of intuitionistic fuzzy sets which deals with both the membership and non-membership degrees of an element belonging to a set. In 2013 [5], Cuong and Kreinovich introduced the theory of picture fuzzy sets

(PFSs) as a generalisation of both fuzzy sets (FSs) and intuitionistic fuzzy sets (IFSs) initiated in [1] and Atanassov [4], respectively. Basically, picture fuzzy sets based models is appropriate in situations involving more answers of type: yes, abstain, no, refusal. A good example of such a situation is voting system in which human voters may decide to: vote for, vote against, abstain and refusal to vote. Thus, according to Cuong and Kreinovich [5], a given set is represented by three membership degrees i.e; positive membership degree, neutral membership degree and negative membership degree.

In 2014, Cuong [6] proposed the distance measure between PFSs. Main fuzzy logic operator on PFSs were investigated by Cuong and Hai [7]. The classification of representable picture t-norm and t-conorm operators for PFSs were obtained by Cuong et al [8]. In 2022 [9], Sangodapo studied the concept of PFSs and obtained some associated properties. The notion of PFSs has been applied by many researchers such as Dutta [10], applied it to medical diagnosis via distance measure between picture fuzzy sets. Rozy and Kaur [11] applied it to medical diagnosis via the similarity measure between picture fuzzy sets. Hasan et al, [12] and [13] applied it in decision making by studying the composition of picture fuzzy relation over picture fuzzy sets.

Yagar in 1986 [14], put forward the notion of fuzzy multisets (FMs). In 2013, Shinoj and Sunil [15] initiated intuitionistic fuzzy multiset (IFMSs), defined some operations on IFMSs and established some of its properties and this was applied in medicine to diagnosis diseases. Due to the fact that the idea of intuitionistic fuzzy multisets also lacks accuracy in handling imprecision and uncertainties because of not taking into account neutrality degree, it is important to study the concept of picture fuzzy multiset as a generalisation of intuitionistic fuzzy multiset.

In [16], Cao et al proposed picture fuzzy multisets (PFMSs) as a generalisation of FM and IFMS, [14] and [15], respectively and also as an extension of PFSs. Sangodapo [17] and [18] introduced picture fuzzy multirelations and some properties related to picture fuzzy multirelations were establshed. In [19], Sangodapo obtained some new algebraic operations and in [20], picture fuzzy multiset was applied to medical diagnosis.

In this paper, some operations on PFSs have been extended to PFMSs. We have proved some theorems that verify these properties. It was shown that the proposed operations are well-defined, internally consistent, and closed under the structure of PFMs. The example demonstrates that the operations handle ambiguity, contradiction, and repetition effectively, making them suitable for applications in multi-criteria decision-making, knowledge representation, and information systems. Furthermore, the concepts of homomorphism and isomorphism theorems in the context of PFMSs were introduced and studied.

2 Preliminaries

In this section, we recall some basic definitions. Throughout, $\mathbb{I}$ denotes the closed interval $[0, 1]$.

Definition 2.1. *[1] Given a nonempty set $\mathcal{C}$. A fuzzy set (FS) $\mathcal{D}$ of $\mathcal{C}$ is written as*

$$\mathcal{D} = \{\langle \frac{\sigma_{\mathcal{D}}(r)}{r} \rangle \mid r \in \mathcal{C}\},$$

with a membership function

$$\sigma_{\mathcal{D}} : \mathcal{C} \longrightarrow \mathbb{I}$$

where the function $\sigma_{\mathcal{D}}(r)$ denotes the degree of membership of $r \in \mathcal{C}$.

Definition 2.2. *[14] A multiset or bag A drawn from a set X is characterised by a function $Count_A$ such that $Count_A : X \to N$ defined by $C_A(x) = n \in \mathbb{N}$ where N is the set of non-negative integers.*

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This multiset theory is a generalization of crisp set theory in which elements are allowed to occur more than once. Unlike in crisp sets, where each element is either present or not (with a membership value of 0 or 1), multiset theory assigns to each element a multiplicity, indicating the number of times that element appears in the collection.

Definition 2.3. *[14] A fuzzy multiset (FMS) $\mathcal{D}$ drawn from $\mathcal{C}$ is characterised by a count membership function $cm_{\mathcal{D}}$ such that $cm_{\mathcal{D}} : \mathcal{C} \to \mathcal{N}$, where $\mathcal{N}$ is the set of all crisp multisets drawn from $\mathbb{I}$. Then, for any $r \in \mathcal{C}$, the value $cm_{\mathcal{D}}(r)$ is a crisp multiset drawn from $\mathbb{I}$. For any $r \in \mathcal{C}$, the membership sequence is defined as the decreasingly ordered sequence of elements in $cm_{\mathcal{D}}(r)$. It is denoted by $(\sigma^1_{\mathcal{D}}(r), \sigma^2_{\mathcal{D}}(r), \cdots, \sigma^d_{\mathcal{D}}(r))$ where $\sigma^1_{\mathcal{D}}(r) \geq \sigma^2_{\mathcal{D}}(r) \geq \cdots \geq \sigma^d_{\mathcal{D}}(r)$.*

Definition 2.4. *[5] Given a nonempty set $\mathcal{C}$. A picture fuzzy set (PFS) $\mathcal{D}$ of $\mathcal{C}$ is written as*

$$\mathcal{D} = \{\langle \frac{\sigma_{\mathcal{D}}(r), \tau_{\mathcal{D}}(r), \gamma_{\mathcal{D}}(r)}{r} \rangle \mid r \in \mathcal{C}\},$$

where the functions

$$\sigma_{\mathcal{D}}(r), \tau_{\mathcal{D}}(r), \gamma_{\mathcal{D}}(r) : \mathcal{C} \to [0,1],$$

are called the positive, neutral and negative membership degrees of $r \in \mathcal{C}$ to $\mathcal{D}$, and for all element $r \in \mathcal{C}$,

$$0 \leq \sigma_{\mathcal{D}}(r) + \tau_{\mathcal{D}}(r) + \gamma_{\mathcal{D}}(r) \leq 1.$$

For each PFS $\mathcal{D}$ of $\mathcal{C}$,

$$\pi_{\mathcal{D}}(r) = 1 - (\sigma_{\mathcal{D}}(r) + \tau_{\mathcal{D}}(r) + \gamma_{\mathcal{D}}(r))$$

is the refusal membership degree of $r \in \mathcal{C}$.

Definition 2.5. *[16] Given a nonempty set $\mathcal{C}$. The picture fuzzy multiset (PFMS) $\mathcal{D}$ in $\mathcal{C}$ is characterised by three functions namely positive membership count function pmc, neutral membership count function $n_e mc$ and negative membership count function nmc such that $pmc, n_e mc, nmc : \mathcal{C} \to \mathcal{N}$, where $\mathcal{N}$, is refer to collection of crisp multisets taken from $\mathbb{I}$. Thus, every element $r \in \mathcal{C}$, pmc is the crisp multiset from $\mathbb{I}$ whose positive membership sequence is defined by $(\sigma^1_{\mathcal{D}}(r), \sigma^2_{\mathcal{D}}(r), \cdots, \sigma^n_{\mathcal{D}}(r))$ such that $\sigma^1_{\mathcal{D}}(r) \geq \sigma^2_{\mathcal{D}}(r) \geq \cdots \geq \sigma^n_{\mathcal{D}}(r)$, $n_e mc$ is the crisp multiset from $\mathbb{I}$ whose neutral membership sequence is defined by $(\tau^1_{\mathcal{D}}(r), \tau^2_{\mathcal{D}}(r), \cdots, \tau^n_{\mathcal{D}}(r))$ and nmc is the crisp multiset from $\mathbb{I}$ whose negative membership sequence is defined by $(\eta^1_{\mathcal{D}}(r), \eta^2_{\mathcal{D}}(r), \cdots, \eta^n_{\mathcal{D}}(r))$, these can be either decreasing or increasing functions satisfying $0 \leq \sigma^k_{\mathcal{D}}(r) + \tau^k_{\mathcal{D}}(r) + \eta^k_{\mathcal{D}}(r) \leq 1 \; \forall r \in \mathcal{C}, \;\; k = 1, 2, \cdots, n$.*

Thus, $\mathcal{D}$ is represented by

$$\mathcal{D} = \{\langle \frac{\sigma^k_{\mathcal{D}}(r), \tau^k_{\mathcal{D}}(r), \eta^k_{\mathcal{D}}(r)}{r} \rangle \mid r \in \mathcal{C}\},$$

$k = 1, 2, \cdots, n$.

The set of all picture fuzzy multisets over $\mathcal{C}$, is denoted as PFMS($\mathcal{C}$).

Operations on picture fuzzy multisets

[16]
Let $\mathcal{D}, \; \mathcal{E} \in$ PFMS($\mathcal{C}$). That is;

$$\mathcal{D} = \{\langle r, \sigma^k_{\mathcal{D}}(r), \tau^k_{\mathcal{D}}(r), \eta^k_{\mathcal{D}}(r)\rangle \mid r \in \; \mathcal{C}\}$$

and

$$\mathcal{E} = \{\langle r, \sigma^k_{\mathcal{E}}(r), \tau^k_{\mathcal{E}}(r), \eta^k_{\mathcal{E}}(r)\rangle \mid r \in \; \mathcal{C}\},$$

where $k = 1, 2, \cdots, n$. Then, the following operations hold:

- $\mathcal{D} \subseteq \mathcal{E}$ if and only if, $\sigma^k_{\mathcal{D}}(r) \leq \sigma^k_{\mathcal{E}}(r)$, $\tau^k_{\mathcal{D}}(r) \leq \tau^k_{\mathcal{E}}(r)$ and $\eta^k_{\mathcal{D}}(r) \geq \eta^k_{\mathcal{E}}(r)$,
- $\mathcal{D} = \mathcal{E}$ if and only if $\mathcal{D} \subseteq \mathcal{E}$ and $\mathcal{E} \subseteq \mathcal{D}$,
- $\mathcal{D} \cup \mathcal{E} = \{(r, \max(\sigma^k_{\mathcal{D}}(r),\ \sigma^k_{\mathcal{E}}(r)), \min(\tau^k_{\mathcal{D}}(r),\ \tau^k_{\mathcal{E}}(r)), \min(\eta^k_{\mathcal{D}}(r),\ \eta^k_{\mathcal{E}}(r))) \mid r \in\ \mathcal{C}\}$,
- $\mathcal{D} \cap \mathcal{E} = \{(r, \min(\sigma^k_{\mathcal{D}}(r),\ \sigma^k_{\mathcal{E}}(r)),\ \max(\tau^k_{\mathcal{D}}(r),\ \tau^k_{\mathcal{E}}(r)),\ \max(\eta^k_{\mathcal{D}}(r),\ \eta^k_{\mathcal{E}}(r))) \mid r \in\ \mathcal{C}\}$,
- $\overline{\mathcal{D}} = \{(r, \eta^k_{\mathcal{D}}(r), \tau^k_{\mathcal{D}}(r), \sigma^k_{\mathcal{D}}(r)) \mid r \in\ \mathcal{C}\}$,
- $\mathcal{D} \times \mathcal{E} = \{\langle (r_1, r_2), \sigma^k_{\mathcal{D}}(r_1) \wedge \sigma^k_{\mathcal{E}}(r_2),\ \tau^k_{\mathcal{D}}(r_1) \wedge \tau^k_{\mathcal{E}}(r_2),\ \eta^k_{\mathcal{D}}(r_1) \vee \eta^k_{\mathcal{E}}(r_2)\rangle \mid r_1, r_2 \in\ \mathcal{C}\}$,
- $\mathcal{D} \oplus \mathcal{E} = \{(r, \sigma^k_{\mathcal{D}}(r) + \sigma^k_{\mathcal{E}}(r) - \sigma^k_{\mathcal{D}}(r)\sigma^k_{\mathcal{E}}(r), \tau^k_{\mathcal{D}}(r)\tau^k_{\mathcal{E}}(r), \eta^k_{\mathcal{D}}(r)\eta^k_{\mathcal{E}}(r)) \mid r \in \mathcal{C}\}$,
- $\mathcal{D} \otimes \mathcal{E} = \{(r, \sigma^k_{\mathcal{D}}(r)\sigma^k_{\mathcal{E}}(r), \tau^k_{\mathcal{D}}(r) + \tau^k_{\mathcal{E}}(r) - \tau^k_{\mathcal{D}}(r)\tau^k_{\mathcal{E}}(r), \eta^k_{\mathcal{D}}(r) + \eta^k_{\mathcal{E}}(r) - \eta^k_{\mathcal{D}}(r)\eta^k_{\mathcal{E}}(r)) \mid r \in \mathcal{C}\}$.

Algebraic laws in picture fuzzy multisets

[16]
For every $\mathcal{D}, \mathcal{E}, \mathcal{F} \in PFMS(\mathcal{C})$. Then the following algebraic laws hold;

1. Involution:
 $\overline{\overline{\mathcal{D}}} = \mathcal{D}$.
2. Commutative Rule:
 (i) $\mathcal{D} \cap \mathcal{E} = \mathcal{E} \cap \mathcal{D}$,
 (ii) $\mathcal{D} \cup \mathcal{E} = \mathcal{E} \cup \mathcal{D}$,
 (iii) $\mathcal{D} \times \mathcal{E} = \mathcal{E} \times \mathcal{D}$.
3. Associative Rule:
 (i) $\mathcal{D} \cap (\mathcal{E} \cap \mathcal{F}) = (\mathcal{D} \cap \mathcal{E}) \cap \mathcal{F}$,
 (ii) $\mathcal{D} \cup (\mathcal{E} \cup \mathcal{F}) = (\mathcal{D} \cup \mathcal{E}) \cup \mathcal{F}$
 (iii) $(\mathcal{D} \times \mathcal{E}) \times \mathcal{F} = \mathcal{D} \times (\mathcal{E} \times \mathcal{F})$.
4. Distributive Rule:
 (i) $\mathcal{D} \cap (\mathcal{E} \cup \mathcal{F}) = (\mathcal{D} \cap \mathcal{E}) \cup (\mathcal{D} \cap \mathcal{F})$,
 (ii) $\mathcal{D} \cup (\mathcal{E} \cap \mathcal{F}) = (\mathcal{D} \cup \mathcal{E}) \cap (\mathcal{D} \cup \mathcal{F})$,
 (iii) $\mathcal{D} \times (\mathcal{E} \cup \mathcal{F}) = (\mathcal{D} \times \mathcal{E}) \cup (\mathcal{D} \times \mathcal{F})$.
 (iv) $\mathcal{D} \times (\mathcal{E} \cap \mathcal{F}) = (\mathcal{D} \times \mathcal{E}) \cap (\mathcal{D} \times \mathcal{F})$.
5. Absorption Rule:
 (i) $\mathcal{D} \cap (\mathcal{D} \cup \mathcal{E}) = \mathcal{D}$,
 (ii) $\mathcal{D} \cup (\mathcal{D} \cap \mathcal{E}) = \mathcal{D}$.
6. Idempotent Rule:
 (i) $\mathcal{D} \cap \mathcal{D} = \mathcal{D}$,
 (ii) $\mathcal{D} \cup \mathcal{D} = \mathcal{D}$.

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7. De Morgan's Rule:
   (i) $\overline{\mathcal{D}\cap\mathcal{E}} = \overline{\mathcal{D}}\cup\overline{\mathcal{E}}$,
   (ii) $\overline{\mathcal{D}\cup\mathcal{E}} = \overline{\mathcal{D}}\cap\overline{\mathcal{E}}$.

Definition 2.6. *[19] Let $\mathcal{D},\mathcal{E},\mathcal{F} \in PFMS(\mathcal{C})$. That is*

$$\mathcal{D} = \{\langle r, \sigma^k_{\mathcal{D}}(r), \tau^k_{\mathcal{D}}(r), \eta^k_{\mathcal{D}}(r)\rangle|\ r \in\ \mathcal{C}\},$$

$$\mathcal{E} = \{\langle r, \sigma^k_{\mathcal{E}}(r), \tau^k_{\mathcal{E}}(r), \eta^k_{\mathcal{E}}(r)\rangle|\ r \in\ \mathcal{C}\},$$

and

$$\mathcal{F} = \{\langle r, \sigma^k_{\mathcal{F}}(r), \tau^k_{\mathcal{F}}(r), \eta^k_{\mathcal{F}}(r)\rangle|\ r \in\ \mathcal{C}\}$$

where $k = 1, 2, \cdots, n$.
Then, define addition and multiplication on PFMSs as

$$\mathcal{D}\oplus\mathcal{E} = \{(r, \sigma^k_{\mathcal{D}\oplus\mathcal{E}}(r), \tau^k_{\mathcal{D}\oplus\mathcal{E}}(r), \eta^k_{\mathcal{D}\oplus\mathcal{E}}(r)) \mid r \in \mathcal{C}\}$$

where $\sigma^k_{\mathcal{D}\oplus\mathcal{E}}(r) = \sigma^k_{\mathcal{D}}(r) + \sigma^k_{\mathcal{E}}(r) - \sigma^k_{\mathcal{D}}(r)\sigma^k_{\mathcal{E}}(r)$,
$\tau^k_{\mathcal{D}\oplus\mathcal{E}}(r) = \tau^k_{\mathcal{D}}(r)\tau^k_{\mathcal{E}}(r)$ and
$\eta^k_{\mathcal{D}\oplus\mathcal{E}}(r) = \eta^k_{\mathcal{D}}(r)\eta^k_{\mathcal{E}}(r)$.

$$\mathcal{D}\otimes\mathcal{E} = \{(r, \sigma^k_{\mathcal{D}\otimes\mathcal{E}}(r), \tau^k_{\mathcal{D}\otimes\mathcal{E}}(r), \eta^k_{\mathcal{D}\otimes\mathcal{E}}(r)) \mid r \in \mathcal{C}\}$$

where $\sigma^k_{\mathcal{D}\otimes\mathcal{E}}(r) = \sigma^k_{\mathcal{D}}(r)\sigma^k_{\mathcal{E}}(r)$,
$\tau^k_{\mathcal{D}\otimes\mathcal{E}}(r) = \tau^k_{\mathcal{D}}(r) + \tau^k_{\mathcal{E}}(r) - \tau^k_{\mathcal{D}}(r)\tau^k_{\mathcal{E}}(r)$ and
$\eta^k_{\mathcal{D}\otimes\mathcal{E}}(r) = \eta^k_{\mathcal{D}}(r) + \eta^k_{\mathcal{E}}(r) - \eta^k_{\mathcal{D}}(r)\eta^k_{\mathcal{E}}(r)$.

Theorem 2.7. *[19] Let $\mathcal{D},\mathcal{E},\mathcal{F}$ be in PFMS($\mathcal{C}$). Then,*

(1) $\mathcal{D}\oplus\mathcal{E} = \mathcal{E}\oplus\mathcal{D}$.

(2) $\mathcal{D}\otimes\mathcal{E} = \mathcal{E}\otimes\mathcal{D}$.

(3) $\mathcal{D}\oplus(\mathcal{E}\oplus\mathcal{F}) = (\mathcal{D}\oplus\mathcal{E})\oplus\mathcal{F}$.

(4) $\mathcal{D}\otimes(\mathcal{E}\otimes\mathcal{F}) = (\mathcal{D}\otimes\mathcal{E})\otimes\mathcal{F}$.

(5) $\mathcal{D}\oplus(\mathcal{E}\cup\mathcal{F}) = (\mathcal{D}\oplus\mathcal{E})\cup(\mathcal{D}\oplus\mathcal{F})$.

(6) $\mathcal{D}\oplus(\mathcal{E}\cap\mathcal{F}) = (\mathcal{D}\oplus\mathcal{E})\cap(\mathcal{D}\oplus\mathcal{F})$.

(7) $\mathcal{D}\otimes(\mathcal{E}\cup\mathcal{F}) = (\mathcal{D}\otimes\mathcal{E})\cup(\mathcal{D}\otimes\mathcal{F})$.

(8) $\mathcal{D}\otimes(\mathcal{E}\cap\mathcal{F}) = (\mathcal{D}\otimes\mathcal{E})\cap(\mathcal{D}\otimes\mathcal{F})$.

3 New Operations on Picture Fuzzy Multisets

In this section, we define some new operations on PFMSs, homomorphism and basic isomorphism theorems on PFMSs. Let $\mathcal{D},\ \mathcal{E} \in$ PFMSs($\mathcal{C}$). That is;

$$\mathcal{D} = \{\langle r, \sigma^k_{\mathcal{D}}(r), \tau^k_{\mathcal{D}}(r), \eta^k_{\mathcal{D}}(r)\rangle \mid r \in \mathcal{C}\}$$

and

$$\mathcal{E} = \{\langle r, \sigma^k_{\mathcal{E}}(r), \tau^k_{\mathcal{E}}(r), \eta^k_{\mathcal{E}}(r)\rangle \mid r \in \mathcal{C}\},$$

where $k = 1, 2, \cdots, n$. Then, the following new operations hold:

1. $\mathcal{D}@\mathcal{E} = \{(r, \sigma^k_{\mathcal{D}@\mathcal{E}}(r), \tau^k_{\mathcal{D}@\mathcal{E}}(r), \eta^k_{\mathcal{D}@\mathcal{E}}(r)) \mid r \in \mathcal{C}\}$,
   where

   $$\sigma^k_{\mathcal{D}@\mathcal{E}}(r) = \frac{1}{2}(\sigma^k_{\mathcal{D}}(r) + \sigma^k_{\mathcal{E}}(r)),\ \tau^k_{\mathcal{D}@\mathcal{E}}(r) = \frac{1}{2}(\tau^k_{\mathcal{D}}(r) + \tau^k_{\mathcal{E}}(r)) \text{ and } \eta^k_{\mathcal{D}@\mathcal{E}}(r)) = \frac{1}{2}(\eta^k_{\mathcal{D}}(r) + \eta^k_{\mathcal{E}}(r)).$$

   The operation is called Arithmetic Mean.

2. $\mathcal{D}\$\mathcal{E} = \{(r, \sigma^k_{\mathcal{D}\$\mathcal{E}}(r), \tau^k_{\mathcal{D}\$\mathcal{E}}(r), \eta^k_{\mathcal{D}\$\mathcal{E}}(r)) \mid r \in \mathcal{C}\}$,
   where

   $$\sigma^k_{\mathcal{D}\$\mathcal{E}}(r) = \sqrt{\sigma^k_{\mathcal{D}}(r).\sigma^k_{\mathcal{E}}(r)},\ \tau^k_{\mathcal{D}\$\mathcal{E}}(r) = \sqrt{\tau^k_{\mathcal{D}}(r).\tau^k_{\mathcal{E}}(r)} \text{ and } \eta^k_{\mathcal{D}\$\mathcal{E}}(r) = \sqrt{\eta^k_{\mathcal{D}}(r).\eta^k_{\mathcal{E}}(r)}.$$

   The operation is called Geometric Mean.

3. $\mathcal{D}\#\mathcal{E} = \{(r, \sigma^k_{\mathcal{D}\#\mathcal{E}}(r), \tau^k_{\mathcal{D}\#\mathcal{E}}(r), \eta^k_{\mathcal{D}\#\mathcal{E}}(r)) \mid r \in \mathcal{C}\}$,
   where

   $$\sigma^k_{\mathcal{D}\#\mathcal{E}}(r) = \frac{2\sigma^k_{\mathcal{D}}(r).\sigma^k_{\mathcal{E}}(r)}{\sigma^k_{\mathcal{D}}(r) + \sigma^k_{\mathcal{E}}(r)},\ \tau^k_{\mathcal{D}\#\mathcal{E}}(r) = \frac{2\tau^k_{\mathcal{D}}(r).\tau^k_{\mathcal{E}}(r)}{\tau^k_{\mathcal{D}}(r) + \tau^k_{\mathcal{E}}(r)} \text{ and } \eta^k_{\mathcal{D}\#\mathcal{E}}(r) = \frac{2\eta^k_{\mathcal{D}}(r).\eta^k_{\mathcal{E}}(r)}{\eta^k_{\mathcal{D}}(r) + \eta^k_{\mathcal{E}}(r)}.$$

   The operation is called Harmonic Mean.

4. $\mathcal{D} * \mathcal{E} = \{(r, \sigma^k_{\mathcal{D}*\mathcal{E}}(r), \tau^k_{\mathcal{D}*\mathcal{E}}(r), \eta^k_{\mathcal{D}*\mathcal{E}}(r)) \mid r \in \mathcal{C}\}$,
   where

   $$\sigma^k_{\mathcal{D}*\mathcal{E}}(r) = \frac{\sigma^k_{\mathcal{D}}(r) + \sigma^k_{\mathcal{E}}(r)}{2(\sigma^k_{\mathcal{D}}(r) + \sigma^k_{\mathcal{E}}(r) + 1)},\ \tau^k_{\mathcal{D}*\mathcal{E}}(r) = \frac{\tau^k_{\mathcal{D}}(r) + \tau^k_{\mathcal{E}}(r)}{2(\tau^k_{\mathcal{D}}(r) + \tau^k_{\mathcal{E}}(r) + 1)} \text{ and } \eta^k_{\mathcal{D}*\mathcal{E}}(r) = \frac{\eta^k_{\mathcal{D}}(r) + \eta^k_{\mathcal{E}}(r)}{2(\eta^k_{\mathcal{D}}(r) + \eta^k_{\mathcal{E}}(r) + 1)}.$$

   The operation is called Normalised Arithmetic Mean.

Example 3.1. *Let* $X = \{a, b\}$,

$$\mathcal{D} = \{(a, 0.70, 0.20, 0.10)(b, 0.60, 0.20, 0.20)\},$$

$$\mathcal{E} = \{(a, 0.50, 0.30, 0.20)(b, 0.70, 0.20, 0.10)\}$$

and

$$\mathcal{F} = \{(a, 0.80, 0.10, 0.10)(b, 0.40, 0.40, 0.20)\}.$$

Then,

$$\mathcal{D}@\mathcal{E} = \{(a, 0.60, 0.25, 0.15), (b, 0.65, 0.20, 0.15)\}$$
$$\mathcal{D}\$\mathcal{E} = \{(a, 0.59, 0.24, 0.14), (b, 0.648, 0.20, 0.14)\}$$
$$\mathcal{D}\#\mathcal{E} = \{(a, 0.58, 0.24, 0.133), (b, 0.646, 0.20, 0.133)\}$$
$$\mathcal{D} * \mathcal{E} = \{(a, 0.44, 0.28, 0.15), (b, 0.46, 0.19, 0.15)\}$$

Theorem 3.2. *Let* $\mathcal{D}, \mathcal{E}$ *and* $\mathcal{F} \in PFMSs(\mathcal{C})$. *Then,*

A. (a) $\mathcal{D}@\mathcal{E} = \mathcal{E}@\mathcal{D}$ *(b)* $\mathcal{D}\$\mathcal{E} = \mathcal{E}\$\mathcal{D}$ *(c)* $\mathcal{D}\#\mathcal{E} = \mathcal{E}\#\mathcal{D}$ *(d)* $\mathcal{D} * \mathcal{E} = \mathcal{E} * \mathcal{D}$

B. (a) $\overline{\mathcal{D}@\mathcal{E}} = \mathcal{D}@\mathcal{E}$ *(b)* $\overline{\mathcal{D}\$\mathcal{E}} = \mathcal{D}\$\mathcal{E}$ *(c)* $\overline{\mathcal{D}\#\mathcal{E}} = \mathcal{D}\#\mathcal{E}$ *(d)* $\overline{\mathcal{D} * \mathcal{E}} = \mathcal{D} * \mathcal{E}$

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Proof. A. (a)

$$\begin{aligned}
\mathcal{D}@\mathcal{E} &= \{(r, \sigma^k_{\mathcal{D}@\mathcal{E}}(r), \tau^k_{\mathcal{D}@\mathcal{E}}(r), \eta^k_{\mathcal{D}@\mathcal{E}}(r)) \mid r \in \mathcal{C}\} \\
&= \{(r, \frac{\sigma^k_{\mathcal{D}}(r) + \sigma^k_{\mathcal{E}}(r)}{2}, \frac{\tau^k_{\mathcal{D}}(r) + \tau^k_{\mathcal{E}}(r)}{2}, \frac{\eta^k_{\mathcal{D}}(r) + \eta^k_{\mathcal{E}}(r)}{2}) \mid r \in \mathcal{C}\} \\
&= \{(r, \frac{\sigma^k_{\mathcal{E}}(r) + \sigma^k_{\mathcal{D}}(r)}{2}, \frac{\tau^k_{\mathcal{E}}(r) + \tau^k_{\mathcal{D}}(r)}{2}, \frac{\eta^k_{\mathcal{E}}(r) + \eta^k_{\mathcal{D}}(r)}{2}) \mid r \in \mathcal{C}\} \\
&= \{(r, \sigma^k_{\mathcal{E}@\mathcal{D}}(r), \tau^k_{\mathcal{E}@\mathcal{D}}(r), \eta^k_{\mathcal{E}@\mathcal{D}}(r)) \mid r \in \mathcal{C}\} \\
&= \mathcal{E}@\mathcal{D}
\end{aligned}$$

(b)

$$\begin{aligned}
\mathcal{D}\$\mathcal{E} &= \{(r, \sigma^k_{\mathcal{D}\$\mathcal{E}}(r), \tau^k_{\mathcal{D}\$\mathcal{E}}(r), \eta^k_{\mathcal{D}\$\mathcal{E}}(r)) \mid r \in \mathcal{C}\} \\
&= \{(r, \sqrt{\sigma^k_{\mathcal{D}}(r).\sigma^k_{\mathcal{E}}(r)}, \sqrt{\tau^k_{\mathcal{D}}(r).\tau^k_{\mathcal{E}}(r)}, \sqrt{\eta^k_{\mathcal{D}}(r).\eta^k_{\mathcal{E}}(r)}) \mid r \in \mathcal{C}\} \\
&= \{(r, \sqrt{\sigma^k_{\mathcal{E}}(r).\sigma^k_{\mathcal{D}}(r)}, \sqrt{\tau^k_{\mathcal{E}}(r).\tau^k_{\mathcal{D}}(r)}, \sqrt{\eta^k_{\mathcal{E}}(r).\eta^k_{\mathcal{D}}(r)}) \mid r \in \mathcal{C}\} \\
&= \{(r, \sigma^k_{\mathcal{E}\wp\mathcal{D}}(r), \tau^k_{\mathcal{E}\wp\mathcal{D}}(r), \eta^k_{\mathcal{E}\$\mathcal{D}}(r)) \mid r \in \mathcal{C}\} \\
&= \mathcal{E}\$\mathcal{D}
\end{aligned}$$

(c)

$$\begin{aligned}
\mathcal{D}\#\mathcal{E} &= \{(r, \sigma^k_{\mathcal{D}\#\mathcal{E}}(r), \tau^k_{\mathcal{D}\#\mathcal{E}}(r), \eta^k_{\mathcal{D}\#\mathcal{E}}(r)) \mid r \in \mathcal{C}\} \\
&= \{(r, \frac{2\sigma^k_{\mathcal{D}}(r).\sigma^k_{\mathcal{E}}(r)}{\sigma^k_{\mathcal{D}}(r) + \sigma^k_{\mathcal{E}}(r)}, \frac{2\tau^k_{\mathcal{D}}(r).\tau^k_{\mathcal{E}}(r)}{\tau^k_{\mathcal{D}}(r) + \tau^k_{\mathcal{E}}(r)}, \frac{2\eta^k_{\mathcal{D}}(r).\eta^k_{\mathcal{E}}(r)}{\eta^k_{\mathcal{D}}(r) + \eta^k_{\mathcal{E}}(r)}) \mid r \in \mathcal{C}\} \\
&= \{(r, \frac{2\sigma^k_{\mathcal{E}}(r).\sigma^k_{\mathcal{D}}(r)}{\sigma^k_{\mathcal{E}}(r) + \sigma^k_{\mathcal{D}}(r)}, \frac{2\tau^k_{\mathcal{E}}(r).\tau^k_{\mathcal{D}}(r)}{\tau^k_{\mathcal{E}}(r) + \tau^k_{\mathcal{D}}(r)}, \frac{2\eta^k_{\mathcal{E}}(r).\eta^k_{\mathcal{D}}(r)}{\eta^k_{\mathcal{E}}(r) + \eta^k_{\mathcal{D}}(r)}) \mid r \in \mathcal{C}\} \\
&= \{(r, \sigma^k_{\mathcal{E}\#\mathcal{D}}(r), \tau^k_{\mathcal{E}\#\mathcal{D}}(r), \eta^k_{\mathcal{E}\#\mathcal{D}}(r)) \mid r \in \mathcal{C}\} \\
&= \mathcal{E}\#\mathcal{D}
\end{aligned}$$

(d)

$$\begin{aligned}
\mathcal{D} * \mathcal{E} &= \{(r, \sigma^k_{\mathcal{D}*\mathcal{E}}(r), \tau^k_{\mathcal{D}*\mathcal{E}}(r), \eta^k_{\mathcal{D}*\mathcal{E}}(r)) \mid r \in \mathcal{C}\} \\
&= \{(r, \frac{\sigma^k_{\mathcal{D}}(r) + \sigma^k_{\mathcal{E}}(r)}{2(\sigma^k_{\mathcal{D}}(r) + \sigma^k_{\mathcal{E}}(r) + 1)}, \frac{\tau^k_{\mathcal{D}}(r) + \tau^k_{\mathcal{E}}(r)}{2(\tau^k_{\mathcal{D}}(r) + \tau^k_{\mathcal{E}}(r) + 1)}, \frac{\eta^k_{\mathcal{D}}(r) + \eta^k_{\mathcal{E}}(r)}{2(\eta^k_{\mathcal{D}}(r) + \eta^k_{\mathcal{E}}(r) + 1)})\} \\
&= \{(r, \frac{\sigma^k_{\mathcal{E}}(r) + \sigma^k_{\mathcal{D}}(r)}{2(\sigma^k_{\mathcal{E}}(r) + \sigma^k_{\mathcal{D}}(r) + 1)}, \frac{\tau^k_{\mathcal{E}}(r) + \tau^k_{\mathcal{D}}(r)}{2(\tau^k_{\mathcal{E}}(r) + \tau^k_{\mathcal{D}}(r) + 1)}, \frac{\eta^k_{\mathcal{E}}(r) + \eta^k_{\mathcal{D}}(r)}{2(\eta^k_{\mathcal{E}}(r) + \eta^k_{\mathcal{D}}(r) + 1)})\} \\
&= \{(r, \sigma^k_{\mathcal{E}*\mathcal{D}}(r), \tau^k_{\mathcal{E}*\mathcal{D}}(r), \eta^k_{\mathcal{E}*\mathcal{D}}(r)) \mid r \in \mathcal{C}\} \\
&= \mathcal{E} * \mathcal{D}
\end{aligned}$$

B. (a)

$$\begin{aligned}
\overline{\mathcal{D}@\mathcal{E}} &= \{(r, \eta^k_{\mathcal{D}@\mathcal{E}}(r)), \tau^k_{\mathcal{D}@\mathcal{E}}(r), \sigma^k_{\mathcal{D}@\mathcal{E}}(r) \mid r \in \mathcal{C}\} \\
&= \{(r, \frac{\eta^k_{\mathcal{D}}(r) + \eta^k_{\mathcal{E}}(r)}{2}, \frac{\tau^k_{\mathcal{D}}(r) + \tau^k_{\mathcal{E}}(r)}{2}, \frac{\sigma^k_{\mathcal{D}}(r) + \sigma^k_{\mathcal{E}}(r)}{2}) \mid r \in \mathcal{C}\} \\
&= \{(r, \frac{\sigma^k_{\mathcal{D}}(r) + \sigma^k_{\mathcal{E}}(r)}{2}, \frac{\tau^k_{\mathcal{D}}(r) + \tau^k_{\mathcal{E}}(r)}{2}, \frac{\eta^k_{\mathcal{D}}(r) + \eta^k_{\mathcal{E}}(r)}{2}) \mid r \in \mathcal{C}\} \\
&= \{(r, \sigma^k_{\mathcal{D}@\mathcal{E}}(r), \tau^k_{\mathcal{D}@\mathcal{E}}(r), \eta^k_{\mathcal{D}@\mathcal{E}}(r)) \mid r \in \mathcal{C}\} \\
&= \mathcal{D}@\mathcal{E}
\end{aligned}$$

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(b)

$$\begin{aligned}
\overline{\mathcal{D}\$\mathcal{E}} &= \{(r, \eta^k_{\mathcal{D}\$\mathcal{E}}(r), \tau^k_{\mathcal{D}\wp\mathcal{E}}(r), \sigma^k_{\mathcal{D}\$\mathcal{E}}(r)) \mid r \in \mathcal{C}\} \\
&= \{(r, \sqrt{\eta^k_{\mathcal{D}}(r).\eta^k_{\mathcal{E}}(r)}, \sqrt{\tau^k_{\mathcal{D}}(r).\tau^k_{\mathcal{E}}(r)}, \sqrt{\sigma^k_{\mathcal{D}}(r).\sigma^k_{\mathcal{E}}(r)}) \mid r \in \mathcal{C}\} \\
&= \{(r, \sqrt{\sigma^k_{\mathcal{D}}(r).\sigma^k_{\mathcal{E}}(r)}, \sqrt{\tau^k_{\mathcal{D}}(r).\tau^k_{\mathcal{E}}(r)}, \sqrt{\eta^k_{\mathcal{D}}(r).\eta^k_{\mathcal{E}}(r)}) \mid r \in \mathcal{C}\} \\
&= \{(r, \sigma^k_{\mathcal{D}\$\mathcal{E}}(r), \tau^k_{\mathcal{D}\$\mathcal{E}}(r), \eta^k_{\mathcal{D}\$\mathcal{E}}(r)) \mid r \in \mathcal{C}\} \\
&= \mathcal{D}\$\mathcal{E}
\end{aligned}$$

(c)

$$\begin{aligned}
\overline{\mathcal{D}\#\mathcal{E}} &= \{(r, \eta^k_{\mathcal{D}\#\mathcal{E}}(r), \tau^k_{\mathcal{D}\sharp\mathcal{E}}(r), \sigma^k_{\mathcal{D}\#\mathcal{E}}(r)) \mid r \in \mathcal{C}\} \\
&= \{(r, \frac{2\eta^k_{\mathcal{D}}(r).\eta^k_{\mathcal{E}}(r)}{\eta^k_{\mathcal{D}}(r)+\eta^k_{\mathcal{E}}(r)}, \frac{2\tau^k_{\mathcal{D}}(r).\tau^k_{\mathcal{E}}(r)}{\tau^k_{\mathcal{D}}(r)+\tau^k_{\mathcal{E}}(r)}, \frac{2\sigma^k_{\mathcal{D}}(r).\sigma^k_{\mathcal{E}}(r)}{\sigma^k_{\mathcal{D}}(r)+\sigma^k_{\mathcal{E}}(r)}) \mid r \in \mathcal{C}\} \\
&= \{(r, \frac{2\sigma^k_{\mathcal{D}}(r).\sigma^k_{\mathcal{E}}(r)}{\sigma^k_{\mathcal{D}}(r)+\sigma^k_{\mathcal{E}}(r)}, \frac{2\tau^k_{\mathcal{D}}(r).\tau^k_{\mathcal{E}}(r)}{\tau^k_{\mathcal{D}}(r)+\tau^k_{\mathcal{E}}(r)}, \frac{2\eta^k_{\mathcal{D}}(r).\eta^k_{\mathcal{E}}(r)}{\eta^k_{\mathcal{D}}(r)+\eta^k_{\mathcal{E}}(r)}) \mid r \in \mathcal{C}\} \\
&= \{(r, \sigma^k_{\mathcal{D}\#\mathcal{E}}(r), \tau^k_{\mathcal{D}\#\mathcal{E}}(r), \eta^k_{\mathcal{D}\#\mathcal{E}}(r)) \mid r \in \mathcal{C}\} \\
&= \mathcal{D}\#\mathcal{E}
\end{aligned}$$

(d)

$$\begin{aligned}
\overline{\mathcal{D}*\mathcal{E}} &= \{(r, \eta^k_{\mathcal{D}*\mathcal{E}}(r), \tau^k_{\mathcal{D}*\mathcal{E}}(r), \sigma^k_{\mathcal{D}*\mathcal{E}}(r)) \mid r \in \mathcal{C}\} \\
&= \{(r, \frac{\eta^k_{\mathcal{D}}(r)+\eta^k_{\mathcal{E}}(r)}{2(\eta^k_{\mathcal{D}}(r)+\eta^k_{\mathcal{E}}(r)+1)}, \frac{\tau^k_{\mathcal{D}}(r)+\tau^k_{\mathcal{E}}(r)}{2(\tau^k_{\mathcal{D}}(r)+\tau^k_{\mathcal{E}}(r)+1)}, \frac{\sigma^k_{\mathcal{D}}(r)+\sigma^k_{\mathcal{E}}(r)}{2(\sigma^k_{\mathcal{D}}(r)+\sigma^k_{\mathcal{E}}(r)+1)})\} \\
&= \{(r, \frac{\sigma^k_{\mathcal{D}}(r)+\sigma^k_{\mathcal{E}}(r)}{2(\sigma^k_{\mathcal{D}}(r)+\sigma^k_{\mathcal{E}}(r)+1)}, \frac{\tau^k_{\mathcal{D}}(r)+\tau^k_{\mathcal{E}}(r)}{2(\tau^k_{\mathcal{D}}(r)+\tau^k_{\mathcal{E}}(r)+1)}, \frac{\eta^k_{\mathcal{D}}+\eta^k_{\mathcal{E}}(r)}{2(\eta^k_{\mathcal{D}}(r)+\eta^k_{\mathcal{E}}(r)+1)})\} \\
&= \{(r, \sigma^k_{\mathcal{D}*\mathcal{E}}(r), \tau^k_{\mathcal{D}*\mathcal{E}}(r), \eta^k_{\mathcal{D}*\mathcal{E}}(r)) \mid r \in \mathcal{C}\} \\
&= \mathcal{D}*\mathcal{E}
\end{aligned}$$

□

Example 3.3. *Using Example 3.1,*

$$\begin{aligned}
\overline{\mathcal{D}@\mathcal{E}} &= \{(a, 0.15, 0.25, 0.60)(b, 0.15, 0.20, 0.65)\} \\
&= \{(a, 0.60, 0.25, 0.15)(b, 0.65, 0.20, 0.15)\} \\
&= \mathcal{D}@\mathcal{E}.
\end{aligned}$$

$$\begin{aligned}
\overline{\mathcal{D}\$\mathcal{E}} &= \{(a, 0.14, 0.25, 0.59)(b, 0.14, 0.20, 0.65)\} \\
&= \{(a, 0.59, 0.25, 0.14)(b, 0.65, 0.20, 0.14)\} \\
&= \mathcal{D}\$\mathcal{E}.
\end{aligned}$$

$$\begin{aligned}
\overline{\mathcal{D}\#\mathcal{E}} &= \{(a, 0.13, 0.24, 0.58)(b, 0.13, 0.20, 0.65)\} \\
&= \{(a, 0.58, 0.24, 0.13)(b, 0.65, 0.20, 0.13)\} \\
&= \mathcal{D}\#\mathcal{E}.
\end{aligned}$$

$$\begin{aligned}
\overline{\mathcal{D}*\mathcal{E}} &= \{(a, 0.15, 0.28, 0.44)(b, 0.15, 0.19, 0.46)\} \\
&= \{(a, 0.44, 0.25, 0.15)(b, 0.46, 0.19, 0.15)\} \\
&= \mathcal{D}*\mathcal{E}.
\end{aligned}$$

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Next we to show that the operations @, \$, # and $*$ are not associative, @ is not distributive over $\oplus$ and $\oplus$ is not distributive over \$.

Example 3.4. *Using $\mathcal{D}, \mathcal{E}$ and $\mathcal{F}$ in Example 3.1,*

$$\mathcal{D}@\mathcal{E} = \{(a, 0.60, 0.25, 0.15), (b, 0.65, 0.20, 0.15)\}$$

so,

$$(\mathcal{D}@\mathcal{E})@\mathcal{F} = \{(a, 0.70, 0.175, 0.125), (b, 0.525, 0.30, 0.175)\}$$
$$\mathcal{E}@\mathcal{F} = \{(a, 0.65, 0.20, 0.15)(b, 0.55, 0.30, 0.15)\}$$

so,

$$\mathcal{D}@(\mathcal{E}@\mathcal{E}) = \{(a, 0.6750, 0.20, 0.125), (b, 0.575, 0.25, 0.175)\}.$$

This implies that $(\mathcal{D}@\mathcal{E})@\mathcal{E} \neq \mathcal{D}@(\mathcal{E}@\mathcal{E})$.

$$\mathcal{D}\$\mathcal{E} = \{(a, 0.59, 0.24, 0.14)(b, 0.65, 0.20, 0.14)\}$$

$$(\mathcal{D}\$\mathcal{E})\$\mathcal{F} = \{(a, 0.69, 0.15, 0.12), (b, 0.51, 0.28, 0.17)\}$$
$$\mathcal{E}\$\mathcal{F} = \{(a, 0.63, 0.17, 0.14)(b, 0.53, 0.28, 0.14)\}$$
$$\mathcal{D}\$(\mathcal{E}\$\mathcal{F}) = \{(a, 0.66, 0.18, 0.12), (b, 0.56, 0.24, 0.17)\}$$

Thus, $(\mathcal{D}\$\mathcal{E})\$\mathcal{E} \neq \mathcal{D}\$(\mathcal{E}\$\mathcal{E})$.

$$\mathcal{D}\#\mathcal{E} = \{(a, 0.62, 0.15, 0.13)(b, 0.51, 0.27, 0.13)\}$$

$$(\mathcal{D}\$\mathcal{E})\$\mathcal{F} = \{(a, 0.67, 0.14, 0.11), (b, 0.50, 0.27, 0.16)\}$$
$$\mathcal{E}\$\mathcal{F} = \{(a, 0.62, 0.15, 0.11)(b, 0.51, 0.27, 0.13)\}$$
$$\mathcal{D}\$(\mathcal{E}\$\mathcal{F}) = \{(a, 0.66, 0.17, 0.11), (b, 0.55, 0.23, 0.16)\}$$

Thus, $(\mathcal{D}\#\mathcal{E})\#\mathcal{E} \neq \mathcal{D}\#(\mathcal{E}\#\mathcal{E})$.

$$\mathcal{D} * \mathcal{E} = \{(a, 0.28, 0.17, 0.12)(b, 0.28, 0.14, 0.12)\}$$
$$(\mathcal{D} * \mathcal{E}) * \mathcal{F} = \{(a, 0.26, 0.18, 0.12), (b, 0.20, 0.18, 0.12)\}$$
$$\mathcal{E} * \mathcal{F} = \{(a, 0.28, 0.14, 0.12)(b, 0.26, 0.19, 0.09)\}$$
$$\mathcal{D} * (\mathcal{E} * \mathcal{F}) = \{(a, 0.25, 0.13, 0.09), (b, 0.23, 0.14, 0.12)\}$$

Thus, $(\mathcal{D} * \mathcal{E}) * \mathcal{E} \neq \mathcal{D} * (\mathcal{E} * \mathcal{E})$.

Example 3.5. *Using $\mathcal{D}, \mathcal{E}$ and $\mathcal{F}$ in Example 3.1,*

$$\mathcal{E} \oplus \mathcal{F} = \{(a, 0.90, 0.03, 0.02), (b, 0.82, 0.08, 0.02)$$

$$\mathcal{D}@(\mathcal{E} \oplus \mathcal{F}) = \{(a, 0.80, 0.115, 0.06), (b, 0.71, 0.14, 0.11)$$
$$\mathcal{D}@\mathcal{E} = \{(a, 0.60, 0.25, 0.15), (b, 0.65, 0.20, 0.15)\}$$
$$\mathcal{D}@\mathcal{F} = \{(a, 0.75, 0.15, 0.10), (b, 0.50, 0.30, 0.20)\}$$
$$(\mathcal{D}@\mathcal{E}) \oplus (\mathcal{D}@\mathcal{F}) = \{(a, 0.9, 0.0375, 0.015), (b, 0.825, 0.06, 0.03)\}$$

Hence,

$$\mathcal{D}@(\mathcal{E} \oplus \mathcal{F}) \neq (\mathcal{D}@\mathcal{E}) \oplus (\mathcal{D}@\mathcal{F})$$

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Example 3.6. *Using $\mathcal{D},\mathcal{E}$ and $\mathcal{F}$ in Example 3.1*

$$\mathcal{E}\$\mathcal{F} = \{(a, 0.63, 0.17, 0.14), (b, 0.53, 0.28, 0.14)$$
$$\mathcal{D} \oplus (\mathcal{E}\$\mathcal{F}) = \{(a, 0.889, 0.034, 0.014), (b, 0.812, 0.056, 0.028)$$
$$\mathcal{D} \oplus \mathcal{E} = \{(a, 0.63, 0.17, 0.14), (b, 0.53, 0.28, 0.14)$$
$$\mathcal{D} \oplus \mathcal{F} = \{(a, 0.94, 0.02, 0.01), (b, 0.76, 0.08, 0.04)$$
$$(\mathcal{D} \oplus \mathcal{E})\$(\mathcal{D} \oplus \mathcal{F}) = \{(a, 0.894, 0.0346, 0.0141), (b, 0.8178, 0.0566, 0.0283).$$

Hence,

$$\mathcal{D}@(\mathcal{E} \oplus \mathcal{F}) \neq (\mathcal{D}@\mathcal{E}) \oplus (\mathcal{D}@\mathcal{F})$$

Theorem 3.7. *Let $\mathcal{D},\mathcal{E}$ and $\mathcal{F} \in PFMSs(\mathcal{C})$. Then,*

(1) $\mathcal{D}@(\mathcal{E} \cup \mathcal{F}) = (\mathcal{D}@\mathcal{E}) \cup (\mathcal{D}@\mathcal{F})$, $\mathcal{D}@(\mathcal{E} \cap \mathcal{F}) = (\mathcal{D}@\mathcal{E}) \cap (\mathcal{D}@\mathcal{F})$.

(2) $\mathcal{D}\$(\mathcal{E} \cup \mathcal{F}) = (\mathcal{D}\$\mathcal{E}) \cup (\mathcal{D}\$\mathcal{F})$, $(\mathcal{D}\$(\mathcal{E} \cap \mathcal{F}) = (\mathcal{D}\$\mathcal{E}) \cap (\mathcal{D}\#\mathcal{F})$.

(3) $\mathcal{D}\#(\mathcal{E} \cup \mathcal{F}) = (\mathcal{D}\#\mathcal{E}) \cup (\mathcal{D}\#\mathcal{F})$, $(\mathcal{D}\#(\mathcal{E} \cap \mathcal{F}) = (\mathcal{D}\#\mathcal{E}) \cap (\mathcal{D}\#\mathcal{F})$.

Proof. (1)

$$\begin{aligned}
\mathcal{D}@(\mathcal{E} \cup \mathcal{F}) &= \mathcal{D}@\{(r, (\sigma^k_{\mathcal{E}}(r) \vee \sigma^k_{\mathcal{F}}(r)), (\tau^k_{\mathcal{E}}(r) \wedge \tau^k_{\mathcal{F}}(r)), (\eta^k_{\mathcal{E}}(r) \wedge \eta^k_{\mathcal{F}}(r))) \mid r \in \ \mathcal{C}\} \\
&= \{(r, \frac{1}{2}(\sigma^k_{\mathcal{D}}(r) + (\max(\sigma^k_{\mathcal{E}}(r),\ \sigma^k_{\mathcal{F}}(r)))), \frac{1}{2}(\tau^k_{\mathcal{D}}(r) + \min(\tau^k_{\mathcal{E}}(r),\ \tau^k_{\mathcal{F}}(r))), \\
&\quad \frac{1}{2}(\eta^k_{\mathcal{D}}(r) + (\min(\eta^k_{\mathcal{E}}(r),\ \eta^k_{\mathcal{F}}(r))))) \mid r \in \ \mathcal{C}\} \\
&= \{(r, \frac{1}{2}(\sigma^k_{\mathcal{D}}(r) + \sigma^k_{\mathcal{E}}(r)), \frac{1}{2}(\tau^k_{\mathcal{D}}(r) + \tau^k_{\mathcal{E}}(r)), \frac{1}{2}(\eta^k_{\mathcal{D}}(r) + \eta^k_{\mathcal{E}}(r)))) \mid r \in \ \mathcal{C}\}\} \\
&\cup \ \{(r, \frac{1}{2}(\sigma^k_{\mathcal{D}}(r) + \sigma^k_{\mathcal{F}}(r)), \frac{1}{2}(\tau^k_{\mathcal{D}}(r) + \tau^k_{\mathcal{F}}(r)), \frac{1}{2}(\eta^k_{\mathcal{D}}(r) + \eta^k_{\mathcal{F}}(r))) \mid r \in \ \mathcal{C}\}\} \\
&= (\mathcal{D}@\mathcal{E}) \cup (\mathcal{D}@\mathcal{F}).
\end{aligned}$$

Similarly, we can prove $\mathcal{D}@(\mathcal{E} \cap \mathcal{F}) = (\mathcal{D}@\mathcal{E}) \cap (\mathcal{D}@\mathcal{F})$.

(2)

$$\begin{aligned}
\mathcal{D}\$(\mathcal{E} \cup \mathcal{F}) &= \mathcal{D}\$\{(r, (\sigma^k_{\mathcal{E}}(r) \vee \sigma^k_{\mathcal{F}}(r)), (\tau^k_{\mathcal{E}}(r) \wedge \tau^k_{\mathcal{F}}(r)), (\eta^k_{\mathcal{E}}(r) \wedge \eta^k_{\mathcal{F}}(r))) \mid r \in \ \mathcal{C}\} \\
&= \{(r, (\sigma^k_{\mathcal{D}}(r)(\sigma^k_{\mathcal{E}}(r) \vee \sigma^k_{\mathcal{F}}(r)))^{\frac{1}{2}}, (\tau^k_{\mathcal{D}}(\tau^k_{\mathcal{E}}(r) \wedge \tau^k_{\mathcal{F}}(r)))^{\frac{1}{2}}, (\eta^k_{\mathcal{D}}(r)(\eta^k_{\mathcal{E}}(r) \wedge \eta^k_{\mathcal{F}}(r)))^{\frac{1}{2}}) \mid r \in \mathcal{C}\} \\
&= \{(r, (\sigma^k_{\mathcal{D}}(r)\sigma^k_{\mathcal{E}}(r))^{\frac{1}{2}}, (\tau^k_{\mathcal{D}}(r)\tau^k_{\mathcal{E}}(z),)^{\frac{1}{2}}, (\eta^k_{\mathcal{D}}(r)\eta^k_{\mathcal{E}}(r))^{\frac{1}{2}}) \mid r \in \ \mathcal{C}\} \\
&\cup \ \{(r, (\sigma^k_{\mathcal{D}}(r)\sigma^k_{\mathcal{F}}(r))^{\frac{1}{2}}, (\tau^k_{\mathcal{D}}(r)\tau^k_{\mathcal{F}}(r),)^{\frac{1}{2}}, (\eta^k_{\mathcal{D}}(r)\eta^k_{\mathcal{F}}(r))^{\frac{1}{2}}) \mid r \in \ \mathcal{C}\} \\
&= (\mathcal{D}\$\mathcal{E}) \cup (\mathcal{D}\$\mathcal{F}).
\end{aligned}$$

Similarly, we can prove $\mathcal{D}\$(\mathcal{E} \cap \mathcal{F}) = (\mathcal{D}\$\mathcal{E}) \cap (\mathcal{D}\$\mathcal{F})$.

(3)

$$\begin{aligned}
\mathcal{D}\#(\mathcal{E} \cup \mathcal{F}) &= \mathcal{D}\#\{(r, (\sigma^k_{\mathcal{E}}(r) \vee \sigma^k_{\mathcal{F}}(r)), (\tau^k_{\mathcal{E}}(r) \wedge \tau^k_{\mathcal{F}}(r)), (\eta^k_{\mathcal{E}}(r) \wedge \eta^k_{\mathcal{F}}(r))) \mid r \in \ \mathcal{C}\} \\
&= \{(r, \frac{2\sigma^k_{\mathcal{D}}(r)(\sigma^k_{\mathcal{E}}(r) \vee \sigma^k_{\mathcal{F}}(r))}{\sigma^k_{\mathcal{D}}(r) + (\sigma^k_{\mathcal{E}}(r) \vee \sigma^k_{\mathcal{F}}(r))}, \frac{2\tau^k_{\mathcal{D}}(r)(\tau^k_{\mathcal{E}}(r) \wedge \tau^k_{\mathcal{F}}(r))}{\tau^k_{\mathcal{D}}(r) + (\tau^k_{\mathcal{E}}(r) \wedge \tau^k_{\mathcal{F}}(r))}, \frac{2\eta^k_{\mathcal{D}}(r)(\eta^k_{\mathcal{E}}(r) \wedge \eta^k_{\mathcal{F}}(r))}{\eta^k_{\mathcal{D}}(r) + (\eta^k_{\mathcal{E}}(r) \wedge \eta^k_{\mathcal{F}}(r))}) \mid r \in \ \mathcal{C}\} \\
&= \{(r, \frac{2\sigma^k_{\mathcal{D}}(r)\sigma^k_{\mathcal{E}}(r)}{\sigma^k_{\mathcal{D}}(r) + \sigma^k_{\mathcal{E}}(r)}, \frac{2\tau^k_{\mathcal{D}}(r)\tau^k_{\mathcal{E}}(r)}{\tau^k_{\mathcal{D}}(r) + \tau^k_{\mathcal{E}}(r)}, \frac{2\eta^k_{\mathcal{D}}(r)\eta^k_{\mathcal{E}}(r)}{\eta^k_{\mathcal{D}}(r) + \eta^k_{\mathcal{E}}(r)}) \mid r \in \ \mathcal{C}\} \\
&\cup \ \{(r, \frac{2\sigma^k_{\mathcal{D}}(r)\sigma^k_{\mathcal{F}}(r)}{\sigma^k_{\mathcal{D}}(r) + \sigma^k_{\mathcal{F}}(r)}, \frac{2\tau^k_{\mathcal{D}}(r)\tau^k_{\mathcal{F}}(r)}{\tau^k_{\mathcal{D}}(r) + \tau^k_{\mathcal{F}}(r)}, \frac{2\eta^k_{\mathcal{D}}(r)\eta^k_{\mathcal{F}}(r)}{\eta^k_{\mathcal{D}}(r) + \eta^k_{\mathcal{F}}(r)}) \mid r \in \ \mathcal{C}\} \\
&= (\mathcal{D}\#\mathcal{E}) \cup (\mathcal{D}\#\mathcal{F}).
\end{aligned}$$

Similarly, we can prove $\mathcal{D}\#(\mathcal{E} \cap \mathcal{F}) = (\mathcal{D}\#\mathcal{E}) \cap (\mathcal{D}\#\mathcal{F})$. ∎

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Theorem 3.8. *Let $\mathcal{D},\mathcal{E}$ and $\mathcal{F} \in PFMSs(\mathcal{C})$. Then,*
(1) $\mathcal{D} \oplus (\mathcal{E}@\mathcal{F}) = (\mathcal{D} \oplus \mathcal{E})@(\mathcal{D} \oplus \mathcal{E})$

(2) $\mathcal{D} \otimes (\mathcal{E}@\mathcal{F}) = (\mathcal{D} \otimes \mathcal{E})@(\mathcal{D} \otimes \mathcal{E})$

Proof. (1)

$$\begin{aligned}
\mathcal{D} \oplus (\mathcal{E}@\mathcal{F}) &= \mathcal{D} \oplus \{(r, (\frac{1}{2}\sigma^k_{\mathcal{E}}(r) + \sigma^k_{\mathcal{F}}(r)), (\frac{1}{2}(\tau^k_{\mathcal{E}}(r) + \tau^k_{\mathcal{F}}(r))), (\frac{1}{2}\eta^k_{\mathcal{E}}(r) + \eta^k_{\mathcal{F}}(r))) \mid r \in \ \mathcal{C}\} \\
&= \{(r, \sigma^k_{\mathcal{D}}(r) + \frac{1}{2}(\sigma^k_{\mathcal{E}}(r) + \sigma^k_{\mathcal{F}}(r)) - \frac{1}{2}(\sigma^k_{\mathcal{D}}(r)(\sigma^k_{\mathcal{E}}(r) + \sigma^k_{\mathcal{F}}(r))), \\
&\quad \frac{1}{2}\tau^k_{\mathcal{D}}(r)(\tau^k_{\mathcal{E}}(r) + \tau^k_{\mathcal{F}}(r)), \frac{1}{2}\eta^k_{\mathcal{D}}(r)(\eta^k_{\mathcal{E}}(r) + \eta^k_{\mathcal{F}}(r))) \mid r \in \ \mathcal{C}\} \\
&= \{(r, \frac{1}{2}\sigma^k_{\mathcal{D}}(r) + \frac{1}{2}\sigma^k_{\mathcal{E}}(r) - \frac{1}{2}\sigma^k_{\mathcal{D}}(r)\sigma^k_{\mathcal{E}}(r), \frac{1}{2}\tau^k_{\mathcal{D}}(r)\tau^k_{\mathcal{E}}(r), \frac{1}{2}\eta^k_{\mathcal{D}}(r)\eta^k_{\mathcal{E}}(r)) \mid r \in \ \mathcal{C}\} \\
&@ \ \{(r, \frac{1}{2}\sigma^k_{\mathcal{D}}(r) + \frac{1}{2}\sigma^k_{\mathcal{F}}(r) - \frac{1}{2}\sigma^k_{\mathcal{D}}(r)\sigma^k_{\mathcal{F}}(r), \frac{1}{2}\tau^k_{\mathcal{D}}(r)\tau^k_{\mathcal{F}}(r), \frac{1}{2}\eta^k_{\mathcal{D}}(r)\eta^k_{\mathcal{F}}(r)) \mid r \in \ \mathcal{C}\} \\
&= \frac{1}{2}\{(r, \sigma^k_{\mathcal{D}}(r) + \sigma^k_{\mathcal{E}}(r) - \sigma^k_{\mathcal{D}}(r)\sigma^k_{\mathcal{E}}(r), \tau^k_{\mathcal{D}}(r)\tau^k_{\mathcal{E}}(r), \eta^k_{\mathcal{D}}(r)\eta^k_{\mathcal{E}}(r)) \mid r \in \ \mathcal{C}\} \\
&@ \ \frac{1}{2}\{(r, \sigma^k_{\mathcal{D}}(r) + \sigma^k_{\mathcal{F}}(r) - \frac{1}{2}\sigma^k_{\mathcal{D}}(r)\sigma^k_{\mathcal{F}}(r), \tau^k_{\mathcal{D}}(r)\tau^k_{\mathcal{F}}(r), \eta^k_{\mathcal{D}}(r)\eta^k_{\mathcal{F}}(r)) \mid r \in \ \mathcal{C}\} \\
&= (\mathcal{D} \oplus \mathcal{E})@(\mathcal{D} \oplus \mathcal{F}).
\end{aligned}$$

(2)

$$\begin{aligned}
\mathcal{D} \otimes (\mathcal{E}@\mathcal{F}) &= \mathcal{D} \otimes \{(r, (\frac{1}{2}\sigma^k_{\mathcal{E}}(r) + \sigma^k_{\mathcal{F}}(r)), (\frac{1}{2}(\tau^k_{\mathcal{E}}(r) + \tau^k_{\mathcal{F}}(r))), (\frac{1}{2}\eta^k_{\mathcal{E}}(r) + \eta^k_{\mathcal{F}}(r))) \mid r \in \ \mathcal{C}\} \\
&= \{(r, \frac{1}{2}\sigma^k_{\mathcal{D}}(r)(\sigma^k_{\mathcal{E}}(r) + \sigma^k_{\mathcal{F}}(r)), \tau^k_{\mathcal{D}}(r) + \frac{1}{2}(\tau^k_{\mathcal{E}}(r) + \tau^k_{\mathcal{F}}(r)) - \frac{1}{2}\tau^k_{\mathcal{D}}(r)(\tau^k_{\mathcal{E}}(r) + \tau^k_{\mathcal{F}}(r)), \\
&\quad \eta^k_{\mathcal{D}}(r) + \frac{1}{2}(\eta^k_{\mathcal{E}}(r) + \eta^k_{\mathcal{F}}(r)) - \frac{1}{2}\eta^k_{\mathcal{D}}(r)(\eta^k_{\mathcal{E}}(r) + \eta^k_{\mathcal{F}}(r)) \mid r \in \ \mathcal{C}\} \\
&= \{(r, \frac{1}{2}\sigma^k_{\mathcal{D}}(r)\sigma^k_{\mathcal{E}}(r), \frac{1}{2}(\tau^k_{\mathcal{D}}(r) + \tau^k_{\mathcal{E}}(r)) - \frac{1}{2}(\tau^k_{\mathcal{D}}(r)\tau^k_{\mathcal{E}}(r)), \\
&\quad \frac{1}{2}(\eta^k_{\mathcal{D}}(r) + \eta^k_{\mathcal{E}}(r)) - \frac{1}{2}(\eta^k_{\mathcal{D}}(r)\eta^k_{\mathcal{E}}(r))) \mid r \in \ \mathcal{C}\} \\
&@ \ \{(r, \frac{1}{2}\sigma^k_{\mathcal{D}}(r)\sigma^k_{\mathcal{F}}(r), \frac{1}{2}(\tau^k_{\mathcal{D}}(r) + \tau^k_{\mathcal{F}}(r)) - \frac{1}{2}(\tau^k_{\mathcal{D}}(r)\tau^k_{\mathcal{F}}(r)), \\
&\quad \frac{1}{2}(\eta^k_{\mathcal{D}}(r) + \eta^k_{\mathcal{F}}(r)) - \frac{1}{2}(\eta^k_{\mathcal{D}}(r)\eta^k_{\mathcal{F}}(r))) \mid r \in \ \mathcal{C}\} \\
&= \frac{1}{2}\{(r, \sigma^k_{\mathcal{D}}(r)\sigma^k_{\mathcal{E}}(r), \tau^k_{\mathcal{D}}(r) + \tau^k_{\mathcal{E}}(r) - \tau^k_{\mathcal{D}}(r)\tau^k_{\mathcal{E}}(r), \eta^k_{\mathcal{D}}(r) + \eta^k_{\mathcal{E}}(r) - \eta^k_{\mathcal{D}}(r)\eta^k_{\mathcal{E}}(r)) \mid r \in \ \mathcal{C}\} \\
&@ \ \frac{1}{2}\{(r, \sigma^k_{\mathcal{D}}(r)\sigma^k_{\mathcal{F}}(r), \tau^k_{\mathcal{D}}(r) + \tau^k_{\mathcal{F}}(r) - \tau^k_{\mathcal{D}}(r)\tau^k_{\mathcal{F}}(r), \eta^k_{\mathcal{D}}(r) + \eta^k_{\mathcal{F}}(r) - \eta^k_{\mathcal{D}}(r)\eta^k_{\mathcal{F}}(r)) \mid r \in \ \mathcal{C}\} \\
&= (\mathcal{D} \otimes \mathcal{E})@(\mathcal{D} \otimes \mathcal{F}).
\end{aligned}$$

□

Example 3.9. *From Example 3.1,*

$$\mathcal{D}@\mathcal{E} = \{(a, 0.60, 0.25, 0.15)(b, 0.65, 0.20, 0.15)\},$$

$$\mathcal{D}\$\mathcal{E} = \{(a, 0.59, 0.25, 0.14), (b, 0.648, 0.20, 0.14)\},$$
$$\mathcal{D}\#\mathcal{E} = \{(a, 0.58, 0.24, 0.133), (b, 0.646, 0.20, 0.133)\}$$

and

$$\mathcal{D} * \mathcal{E} = \{(a, 0.44, 0.28, 0.15), (b, 0.46, 0.19, 0.15)\}.$$

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Using the same Example 3.1, we find;

$$\mathcal{E} \cup \mathcal{F} = \{(a, 0.80, 0.10, 0.10), (b, 0.70, 0.20, 0.10)\}$$

$$\mathcal{E} \cap \mathcal{F} = \{(a, 0.50, 0.30, 0.20)(b, 0.40, 0.40, 0.20)\}$$

$$\mathcal{D}@(\mathcal{E} \cup \mathcal{F}) = \{(a, 0.75, 0.15, 0.10), (b, 0.65, 0.20, 0.15)\}$$

$$\mathcal{D}@\mathcal{F} = \{(a, 0.75, 0.15, 0.10), (b, 0.50, 0.30, 0.20)\}$$

Then,

$$(\mathcal{D}@\mathcal{E}) \cup (\mathcal{D}@\mathcal{F}) = \{(a, 0.75, 0.15, 0.10), (b, 0.65, 0.20, 0.15)\}$$

$$\Rightarrow \ \mathcal{D}@(\mathcal{E} \cup \mathcal{F}) = (\mathcal{D}@\mathcal{E}) \cup (\mathcal{D}@\mathcal{F}).$$

$$\mathcal{D}@(\mathcal{E} \cap \mathcal{F}) = \{(a, 0.60, 0.25, 0.15), (b, 0.50, 0.30, 0.20)\}$$

$$(\mathcal{D}@\mathcal{E}) \cap (\mathcal{D}@\mathcal{F}) = \{(a, 0.60, 0.25, 0.15), (b, 0.50, 0.30, 0.20)\}$$

$$\Rightarrow \ \mathcal{D}@(\mathcal{E} \cap \mathcal{F}) = (\mathcal{D}@\mathcal{E}) \cap (\mathcal{D}@\mathcal{F}).$$

$$\mathcal{D}\$(\mathcal{E} \cup \mathcal{F}) = \{(a, 0.75, 0.14, 0.10), (b, 0.65, 0.20, 0.14)\}$$

$$\mathcal{D}\$\mathcal{E} = \{(a, 0.59, 0.25, 0.14), (b, 0.648, 0.20, 0.14)\}$$

$$\mathcal{D}\$\mathcal{F} = \{(a, 0.75, 0.14, 0.10), (b, 0.50, 0.28, 0.20)\}$$

Then,

$$(\mathcal{D}\$\mathcal{E}) \cup (\mathcal{D}\$\mathcal{F}) = \{(a, 0.75, 0.14, 0.10), (b, 0.65, 0.20, 0.14)\}$$

$$\Rightarrow \ \mathcal{D}\$(\mathcal{E} \cup \mathcal{F}) = (\mathcal{D}\$\mathcal{E}) \cup (\mathcal{D}\$\mathcal{F}).$$

$$\mathcal{D}\$(\mathcal{E} \cap \mathcal{F}) = \{(a, 0.59, 0.25, 0.14), (b, 0.50, 0.28, 0.2)\}$$

$$(\mathcal{D}\$\mathcal{E}) \cap (\mathcal{D}\$\mathcal{F}) = \{(a, 0.59, 0.25, 0.14), (b, 0.50, 0.28, 0.2)\}$$

$$\Rightarrow \ \mathcal{D}\$(\mathcal{E} \cap \mathcal{F}) = (\mathcal{D}\$\mathcal{E}) \cap (\mathcal{D}\$\mathcal{F}).$$

$$\mathcal{D}\#(\mathcal{E} \cup \mathcal{F}) = \{(a, 0.75, 0.13, 0.10), (b, 0.65, 0.20, 0.13)\}$$

$$\mathcal{D}\#\mathcal{E} = \{(a, 0.58, 0.24, 0.13), (b, 0.65, 0.20, 0.13)\}$$

$$\mathcal{D}\#\mathcal{F} = \{(a, 0.75, 0.13, 0.10), (b, 0.48, 0.27, 0.20)\}$$

Then,

$$(\mathcal{D}\#\mathcal{E}) \cup (\mathcal{D}\#\mathcal{F}) = \{(a, 0.75, 0.13, 0.10), (b, 0.65, 0.20, 0.13)\}$$

$$\Rightarrow \ \mathcal{D}\#(\mathcal{E} \cup \mathcal{F}) = (\mathcal{D}\#\mathcal{E}) \cup (\mathcal{D}\#\mathcal{F}).$$

$$\mathcal{D}\#(\mathcal{E} \cap \mathcal{F}) = \{(a, 0.58, 0.24, 0.13), (b, 0.48, 0.27, 0.20)\}$$

$$(\mathcal{D}\#\mathcal{E}) \cap (\mathcal{D}\#\mathcal{F}) = \{(a, 0.58, 0.24, 0.13), (b, 0.48, 0.27, 0.20)\}$$

$$\Rightarrow \ \mathcal{D}\#(\mathcal{E} \cap \mathcal{F}) = (\mathcal{D}\#\mathcal{E}) \cap (\mathcal{D}\#\mathcal{F}).$$

$$\mathcal{D}@(\mathcal{E} \cap \mathcal{F}) = \{(a, 0.60, 0.25, 0.15), (b, 0.50, 0.30, 0.20)\}$$

Then,

$$(\mathcal{D}@\mathcal{E}) \cap (\mathcal{D}@\mathcal{F}) = \{(a, 0.60, 0.25, 0.15), (b, 0.50, 0.30, 0.20)\}$$

$$\Rightarrow \mathcal{D}@(\mathcal{E} \cup \mathcal{F}) = (\mathcal{D}@\mathcal{E}) \cup (\mathcal{D}@\mathcal{F}).$$

$$\mathcal{E}@\mathcal{F} = \{(a, 0.65, 0.20, 0.15)(b, 0.55, 0.30, 0.15)\}$$

Then,

$$\mathcal{D} \oplus (\mathcal{E}@\mathcal{F}) = \{(a, 0.895, 0.360, 0.235)(b, 0.820, 0.440, 0.320)\}$$
$$\mathcal{D} \oplus \mathcal{E} = \{(a, 0.85, 0.44, 0.28)(b, 0.88, 0.36, 0.28)\}$$
$$\mathcal{D} \oplus \mathcal{F} = \{(a, 0.94, 0.28, 0.19)(b, 0.76, 0.52, 0.36)\}$$

Then,

$$(\mathcal{D} \oplus \mathcal{E})@(\mathcal{D} \oplus \mathcal{F}) = \{(a, 0.895, 0.360, 0.235)(b, 0.820, 0.440, 0.320)\}$$
$$\Rightarrow \mathcal{D} \oplus (\mathcal{E}@\mathcal{F}) = (\mathcal{D} \oplus \mathcal{E})@(\mathcal{D} \oplus \mathcal{F}).$$

$$\mathcal{D} \otimes (\mathcal{E}@\mathcal{F}) = \{(a, 0.455, 0.360, 0.235)(b, 0.330, 0.440, 0.320)\}$$
$$\mathcal{D} \otimes \mathcal{E} = \{(a, 0.35, 0.44, 0.28)(b, 0.42, 0.36, 0.28)\}$$
$$\mathcal{D} \otimes \mathcal{F} = \{(a, 0.56, 0.28, 0.19)(b, 0.24, 0.52, 0.36)\}$$

Then,

$$(\mathcal{D} \otimes \mathcal{E})@(\mathcal{D} \otimes \mathcal{F}) = \{(a, 0.455, 0.360, 0.235)(b, 0.330, 0.440, 0.320)\}$$
$$\Rightarrow \mathcal{D} \otimes (\mathcal{E}@\mathcal{F}) = (\mathcal{D} \otimes \mathcal{E})@(\mathcal{D} \otimes \mathcal{F}).$$

Definition 3.10. *Let*

$$\mathcal{P} = \{\langle r, \sigma^k_{\mathcal{P}}(r), \tau^k_{\mathcal{P}}(r), \eta^k_{\mathcal{P}}(r)\rangle \mid r \in \mathcal{C}\}$$

and

$$\mathcal{Q} = \{\langle r, \sigma^k_{\mathcal{Q}}(r), \tau^k_{\mathcal{Q}}(r), \eta^k_{\mathcal{Q}}(r)\rangle \mid r \in \mathcal{C}\},$$

*where $k = 1, 2, \cdots, n$. be PFMSs. Then, a mapping $f : \mathcal{P} \to \mathcal{Q}$ is called a homomorphism under the operation $\triangle \in \{@, \$, \#, *\}$, if $f(\mathcal{D}\triangle\mathcal{E}) = f(\mathcal{D})\triangle f(\mathcal{D}),\ \forall\ \mathcal{D},\ \mathcal{E}\ \in \mathcal{P}$*

Remark 3.11. *If $\triangle = @,\ \$,\ \#,\ *$, then f is a homomorphism under $@,\ \$,\ \#,\ *$.*

Example 3.12. *From Example 3.1,*

$$\mathcal{D}@\mathcal{E} = \{(a, 0.60, 0.25, 0.15)(b, 0.65, 0.20, 0.15)\},$$
$$\mathcal{D}\$\mathcal{E} = \{(a, 0.59, 0.25, 0.14), (b, 0.648, 0.20, 0.14)\},$$
$$\mathcal{D}\#\mathcal{E} = \{(a, 0.58, 0.24, 0.133), (b, 0.646, 0.20, 0.133)\}$$

and

$$\mathcal{D} * \mathcal{E} = \{(a, 0.44, 0.28, 0.15), (b, 0.46, 0.19, 0.15)\}.$$

Define $f : \mathcal{P} \to \mathcal{Q}$ as $f(\mathcal{R})(r) = \frac{1}{2}(\sigma^k_{\mathcal{R}}(r), \tau^k_{\mathcal{R}}(r), \eta^k_{\mathcal{R}}(r))$. Thus,

$$f(\mathcal{D}) = \{(a, 0.35, 0.10, 0.05)(b, 0.30, 0.10, 0.10)\}$$

and

$$f(\mathcal{E}) = \{(a, 0.25, 0.15, 0.10)(b, 0.35, 0.10, 0.05)\}$$

So,

$$f(\mathcal{D})@f(\mathcal{E}) = \{(a, 0.30, 0.125, 0.075)(b, 0.325, 0.10, 0.075)\}$$

and

$$f(\mathcal{D}@\mathcal{E}) = \{(a, 0.30, 0.125, 0.075)(b, 0.325, 0.10, 0.075)\}$$

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Thus, $f(\mathcal{D}@\mathcal{E}) = f(\mathcal{D})@f(\mathcal{D})$.

$$f(\mathcal{D}\$\mathcal{E}) = \{(a, 0.295, 0.122, 0.071)(b, 0.324, 0.10, 0.071)\}.$$

$$f(\mathcal{D})\$f(\mathcal{E}) = \{(a, 0.295, 0.122, 0.071)(b, 0.324, 0.10, 0.071)\}.$$

Thus, $f(\mathcal{D}\$\mathcal{E}) = f(\mathcal{D})\$f(\mathcal{D})$.

$$f(\mathcal{D}\$\mathcal{E}) = \{(a, 0.29, 0.12, 0.067)(b, 0.323, 0.10, 0.067)\}.$$

So,

$$f(\mathcal{D})\$f(\mathcal{E}) = \{(a, 0.295, 0.122, 0.071)(b, 0.324, 0.10, 0.071)\}.$$

Thus, $f(\mathcal{D}\$\mathcal{E}) = f(\mathcal{D})\$f(\mathcal{D})$.

$$f(\mathcal{D} * \mathcal{E}) = \{(a, 0.22, 0.14, 0.075)(b, 0.23, 0.095, 0.075)\}.$$

So,

$$f(\mathcal{D}) * f(\mathcal{E}) = \{(a, 0.188, 0.10, 0.065)(b, 0.196, 0.083, 0.065)\}.$$

Thus, $f(\mathcal{D} * \mathcal{E}) \neq f(\mathcal{D}) * f(\mathcal{D})$.
Therefore, f *is not a homomorphism under* $*$.

Next, we define first, second and third isomorphism theorems.

Definition 3.13. $ker(f) = \{\mathcal{D}, \mathcal{E} \in \mathcal{P} \mid f(\mathcal{D}) = f(\mathcal{E})\}$ *and* $Im(f) = \{f(\mathcal{D}) \mid \mathcal{D} \in \mathcal{P}\}$.

Definition 3.14. *Let* $\mathcal{P}$ *and* $\mathcal{Q}$ *be two PFMSs over* $\mathcal{C}_1$ *and* $\mathcal{C}_2$*, respectively equipped with a common binary operation* $\triangle \in \{@, \$, \#, *\}$ *where* $\triangle$ *is defined elementwise for picture fuzzy degrees. A mapping* $f : (\mathcal{P}, \triangle) \to (\mathcal{Q}, \triangle)$ *is called a PFMS isomorphism (under the operation* $\triangle$*) if* f *is bijective and* f *preserve the operation, i.e,* $f(\mathcal{D}\triangle\mathcal{E}) = f(\mathcal{D})\triangle f(\mathcal{E})$, $\forall\ \mathcal{D}, \mathcal{E} \in \mathcal{P}$.
If such an isomorphism exists, the PFMS structure $(\mathcal{P}, \triangle)$ *and* $(\mathcal{Q}, \triangle)$ *are said to be isomorphic, i.e,* $(\mathcal{P}, \triangle) \cong (\mathcal{Q}, \triangle)$.

Definition 3.15. *If* $f : \mathcal{D} \to \mathcal{E}$ *is a homomorphism, then first isomorphism states that*

$$\mathcal{D}/ker(f) \cong Im(f).$$

Definition 3.16. *If* $\mathcal{D}, \mathcal{E} \subseteq \mathcal{F}$*, then second isomorphism states that*

$$(\mathcal{D} + \mathcal{E})/\mathcal{E} \cong \mathcal{D}/\mathcal{D} \cap \mathcal{E}.$$

$\mathcal{D} + \mathcal{E} :=$ *componentwise maximum for* σ, τ *and minimum for* η.
$\mathcal{D} \cap \mathcal{E} :=$ *componenitewise minimum for* σ, τ *and maximum for* η.

Definition 3.17. *If* $\mathcal{D} \subseteq \mathcal{E} \subseteq \mathcal{F}$*, then third isomorphism states that*

$$(\mathcal{F}/\mathcal{D})/(\mathcal{E}/\mathcal{D}) \cong \mathcal{F}/\mathcal{E}.$$

Example 3.18. *Let*

$$\mathcal{D} = \{(a, 0.40, 0.10, 0.50)(b, 0.30, 0.10, 0.40)\},$$

$$\mathcal{E} = \{(a, 0.50, 0.20, 0.30)(b, 0.40, 0.20, 0.30)\}$$

and

$$\mathcal{F} = \{(a, 0.60, 0.30, 0.10)(b, 0.50, 0.30, 0.20)\}$$

Define

$$f(\mathcal{R})(r) = \alpha\mathcal{R} = \alpha(\sigma^k_{\mathcal{R}}(r), \tau^k_{\mathcal{R}}(r), \eta^k_{\mathcal{R}}(r)), \quad \alpha = \frac{1}{2}.$$

$f(\mathcal{D}) = \{(a, 0.20, 0.05, 0.25)(b, 0.10, 0.05, 0.20)\}$, *also* $ker(f) = \{f(\mathcal{R}) \mid f(\mathcal{R}) = (0, 0, 0)\}$ *since kernel scales everything by* $\lambda = 0.5$, *the quotient recovers the image under the same scaling.*

$$\mathcal{D} + \mathcal{E} = \{(a, 0.50, 0.20, 0.30)(b, 0.40, 0.20, 0.30)\} = \mathcal{E}.$$

So,

$$(\mathcal{D} + \mathcal{E})/\mathcal{E} = \mathcal{E}/\mathcal{E} = \{(a, 0.00, 0.00, 0.00)(b, 0.00, 0.00, 0.00)\}$$

$$\mathcal{D} \cap \mathcal{E} = \{(a, 0.40, 0.10, 0.50)(b, 0.30, 0.10, 0.40)\} = \mathcal{D}.$$

So,

$$(\mathcal{D} \cap \mathcal{E})/\mathcal{D} = \mathcal{D}/\mathcal{D} = \{(a, 0.00, 0.00, 0.00)(b, 0.00, 0.00, 0.00)\}$$

Define $a - b = max(0, a - b)$, *thus,* $\mathcal{F}/\mathcal{D} = \{(a, 0.20, 0.20, 0.20)(b, 0.20, 0.20, 0.20)\}$
$\mathcal{E}/\mathcal{D} = \{(a, 0.10, 0.10, 0.10)(b, 0.10, 0.10, 0.10)\}$
Then,

$$(\mathcal{F}/\mathcal{D})/(\mathcal{E}/\mathcal{D}) = \{(a, 0.10, 0.10, 0.10)(b, 0.10, 0.10, 0.10)\} = \mathcal{E}/\mathcal{D}$$

$$\mathcal{F}/\mathcal{E} = \{(a, 0.10, 0.10, 0.10)(b, 0.10, 0.10, 0.10)\}.$$

4 Discussion

The results established in this work demonstrate that the operations @, \$, #, and $*$ on PFMS satisfy commutativity and are distributive over union and intersection.

A particularly noteworthy result is that both $\oplus$ and $\otimes$ distribute over @. This behavior highlights the linear interaction between structural composition and averaging, allowing for consistent aggregation in layered systems. It also supports modular model design where evaluations or inputs are aggregated at different stages.

Despite these strengths, Example 3.4, Example 3.5 and Example 3.6 confirmed the failure of associativity, distributivity for all operations and distributivity of $\oplus$ and $\otimes$ over \$, # and $*$, respectively. In particular, the operation $*$ also fails to distribute over union and intersection.

Additionally, we have introduced the concept of homomorphism and the basic isomorphism theorems in the context of PFMS.

5 Conclusion

In this paper, we proposed some new operations @, \$, # and $*$ for picture fuzzy multisets (PFMS), with the aim of enhancing the algebraic structure and applicability of PFMS in complex uncertain environments. The theoretical results established in this study confirm that the operations @, \$, # and $*$ are commutative and distribute well over key set-theoretic operations. Some theorems based on the new operations were obtained through several algebraic properties including commutativity, associativity and distributivity and examples were given to check the validity of the newly proposed operations. We remarked that;

- The new operations @, \$, # and $*$ violate the associativity law, see Example 3.4,
- The new operations @, \$, # and $*$ violate the distributivity law over $\oplus$ and $\otimes$, see Example 3.5,

- The operations $\oplus$ and $\otimes$ violate distributivity over \$, # and $*$ see Example 3.6 and
- The operations @, \$, # distributive over $\cup$ and $\cap$ and $*$ is not.

Also, some algebraic principles, such as homomorphism and basic isomorphism theorems in the context of the PFMSs were established.

References

[1] Zadeh L. A., (1965), Fuzzy sets, Inform and Control, 8 338-353.

[2] Edeghagba, E. E, Abubakar, S. I, Umar, F. M and Abdullahi. M., (2024), Full fuzzy parametrized soft expert set with application to decision making, International Journal of Mathematical Sciences and Optimization: Theory and Applications, 10(4), 12-28.

[3] Abubakar Umar Terrang, Awumtiya Kumba Isa, Felix Bakare, Patience Bwanu Iliya., (2021), On solution of first order initial value problems using Laplace transform in fuzzy environment. International Journal of Mathematical Sciences and Optimization: Theory and Applications, 2(4), 12-29.

[4] Atanassov K. T., (1986), Intuitionistic fuzzy sets, Fuzzy sets and Systems 20 87-96.

[5] Cuong B. C and Kreinovich V., (2013), Picture Fuzzy Sets-a new concept for Computational intelligence problems, Proceeding of the Third World Congress on Information and Communication Technologies, 1 - 6.

[6] Cuong B. C., (2014), Picture fuzzy sets, Journal of Computer Science and Cybernetics, 30 409-420.

[7] Cuong B. C and Hai P. V., (2015), Some fuzzy logic operators for picture fuzzy sets. Seventh International Conference on Knowledge and Systems Engineering, 132 - 137.

[8] Cuong B. C, Kreinovich V and Ngan R. T., (2016), A classification of representable t-norm operators for picture fuzzy sets, Eighth International Conference on Knowledge and Systems Engineering Vietnam.

[9] Sangodapo T. O., (2022), Some notions of picture fuzzy sets, Journal of the Nigerian Mathematical Society 41(2), 83-103.

[10] Dutta P., (2018), Medical diagnosis based on distance measures between picture fuzzy sets, International Journal of Fuzzy System Applications 7(4), 15-36.

[11] Rozy and Kaur G., (2022), Similarity measure between picture fuzzy sets and its applications in medical diagnosis. Journal of Advances and Scholarly Researches in Allied Education 19(3), 1-3.

[12] Hasan M. K, Sultana A and Mitra N. K., (2023), Composition of picture fuzzy relations with applications in decision making. EJ-MATH, Europian Journal of Mathematics and Statistics 4(2), 19-28.

[13] Hasan M. K, Sultana A and Mitra N. K., (2023), Picture fuzzy relations over picture fuzzy sets, American Journal of Computational Mathematics, 13 161-184.

[14] Yager R. R., (1986), On the Theory of Bags, Int. J. of General System 13 23-37.

[15] Shinoj T. K and Sunil J. J., (2013), Intuitionistic fuzzy multisets, International Journal of Engineering Science and Innovative Technology 2(6), 1-24.

[16] Cao L, Feng Y and Sangodapo T. O., (2023), Picture fuzzy multisets, Italian Journal of Pure and Applied Mathematics, 51 64-76.

[17] Sangodapo T. O., (2024), Some properties of picture fuzzy multirelations, Journal of the Nigerian Mathematical Society, 43(4), 337-354.

[18] Sangodapo T. O., (2025), Some results on picture fuzzy multirelations, Journal of Mathematics and Informatics 28 45-58.

[19] Sangodapo T. O., (2025), New algebraic operations of picture fuzzy multisets, Intern. J. of Fuzzy Mathematical Archive (IJFMA), 23(1), 31-39.

[20] Feng Y and Sangodapo T. O., (2025), Distance measures between picture fuzzy multisets and their application to medical diagnosis, Italian Journal of Pure and Applied Mathematics, Accepted.