

An Improved Block Backward Differentiation Formula for Stiff Differential Equations

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Abstract

In this work, we derived eight-order, implicit block formula using backward differentiation method with two off-step points for solving stiff ordinary differential equations. We solve some standard set of stiff initial value problems (IVPs) using the new method. We then compared the numerical results with the existing methods which have solved the same set of ivps. We have also verified the consistency, order and stability of the method whereby, method is found to be A - stable and consistent.

Keywords: Implicit, Block Backward Differentiation, Stability; Off-Steps, Stiff Differential Equations.

MSC2010: 65L04.

1 Introduction

In science and engineering, mathematical models are often developed to aid in the understanding of physical phenomena. These models often result in equations that involve derivatives of an unknown function of one or more variables. Such equations are called differential equations. Differential equations arise not only in the physical sciences but also in diverse fields such as economics, medicine, psychology, operations research, biology, and anthropology. Interestingly, differential equations arising from the modeling of physical phenomena often do not have analytic solutions. Hence, the development of numerical methods to obtain approximate solutions becomes necessary. That extent, several numerical methods, such as the finite difference method, finite element method, and finite volume method, among others have been developed, depending on the nature and type of differential equation to be solved. A differential equation in which the unknown function depends on two or more independent variables is called a partial differential equation, while one in which the unknown function depends on only one independent variable is called an ordinary differential equation. This

work focuses on the study of numerical solutions of the latter, specifically addressing numerical methods for solving Initial Value Problems (IVPs) of Ordinary Differential Equations (ODEs) of the form

$$y'(x) = f(x, y), \quad y(a) = p, \quad x \in [a, b] \quad (1.1)$$

Generally, the problems represented by equation (1.1) can be classified into two type; non-stiff ordinary differential equations and stiff ordinary differential equations. In non-stiff problems, all components evolve simultaneously and on comparable time-scales. These problems are often effectively solved using explicit methods with appropriate error control. The second type is stiff ordinary differential equations, presents a greater challenge. The term 'stiffness' was first introduced by Curtiss and Hirschfelder (1952) [1] in their study of chemical kinetics. Although, various definitions of stiffness exist in the literature, there is no universally accepted definition. For the purpose of this study, the following definition of stiffness is adopted. A linear system of the form given in equation (1.1) is said to be stiff if

1. $Re(\lambda_i) < 0, \quad i = 1, 2, \dots, d.$
2. $Maxi |Re(\lambda_i)| \gg Mini |Re(\lambda_i)|$, where λ_i are the eigenvalues of A and the ratio $S = \frac{Maxi |Re(\lambda_i)|}{Mini |Re(\lambda_i)|}$ is called the stiffness ratio as a measure of stiffness (Lambert, J. D. (1973).

The consideration of stability properties is particularly important when developing methods for solving stiff ODEs which possess an A-stable. Zakman et.al (2016) [2] proposed an A-stable one-step block method of order four for solving a stiff ordinary differential equation. The method approximate the solutions of a stiff ordinary differential equation at three points simultaneously using a constant step size. The method is similar to the convetional one-step method and it is self-starting, however, its implementation relies on predictor-corrector formulae. In a related development, Rufai et.al (2016) [3] derived one-sixth hybrid block method for the general solution of first order initial values problems of ordinary differential equations. The proposed method was derived by using the approach of collocation and interpolation on Chebyshev polynomials. By approximating the solution at selected points, a continuous linear multi step method was constructed and subsequently evaluated at off-grid points to generate hybrid linear multi step methods. Ukpebor and Adoghe (2019) [4], employed a similar approach to develop the main Continuous Fourth Derivative Method (CFDM), which was then used to obtain additional methods. These were combined to form a single block method.

Noor and Ibrahim (2019) [5] developed an A stable Block Method for the solution of Stiff ordinary differential equations using 2-point Block Backward Differentiation formula (BBDF). Their work resulted into an improved accuracy, compared to conventional methods. Similarly, Bibi (2020) [6] proposed a sixth-order fully implicit BBDF method in corporating two off-step points for the integration of ordinary differential equations. The results demonstrated superior performance when compared with other existing methods in the literature. Ibrahim and Nasarudin (2020) [7] developed a class of Hybrid Block Backward Differentiation Formula (HBBDF) methods, which possess A-stability. These methods were constructed by reformulating the BBDF approach for solving stiff ordinary differential equations. The stability and convergence of the proposed methods were thoroughly investigated. The analysis showed that the methods are zero-stable and consistent, and thus, by the Dahlquist Equivalence Theorem, the methods are convergent. Nasarudin et.al (2020) [8], proposed a sixth-order fully implicit Block Backward Differentiation Formula (BBDF) incorporating two off-step points for the integration of first-order ordinary differential equations that exhibit stiffness. The scheme was shown to be convergent through rigorous analysis. In a related study, Abd Rasid (2021) [9] introduced a novel alternative approach to the implicit diagonal BBDF for solving linear and nonlinear first-order stiff ordinary differential equations. The solver was developed by manipulating the number of back ward values in order to achieve a higher order of accuracy using an interpolation-based procedure. C. Chibuisi et al (2022) [10] implemented second derivative block backward differentiation formulae methods in solving first order delay differential equations without the application of interpolation methods in investigating the delay argument.

The delay argument was evaluated using a suitable idea of sequence which was incorporated in to some first order delay differential equations before its numerical evaluations. B. I. Akinnukawe and E. M. Atteh (2024) [11] proposed a novel block method to solve the nonlinear time dependent Burgers' equation. The Burgers' PDE is semi discretized in spatial direction by using the standard fourth-order compact difference schemes to yield system of nonlinear ordinary differential equations (ODE) in time.

2 Methodology and Results

2.1 Problem Definition

In this paper, we shall consider the numerical solution of the first order initial value problems (IVPs) for ordinary differential equations of the form

$$y'(x) = f(x, y), \quad y(a) = y_0, \quad x \in [a, b] \quad (2.1)$$

Equation (2.1) is said to be linear if $f(x, y) = A(x)y + \Phi(x)$, where $A(x)$ is a constant $d \times d$ matrix and $\phi(x)$ is a d -dimensional vector assumed to be continuously differentiable, and if it satisfies the Lipschitz condition, guarantees the existence and uniqueness of the solution of equation.

2.2 Methodology

For the derivation of our method, we consider the following block with step size h as shown below:

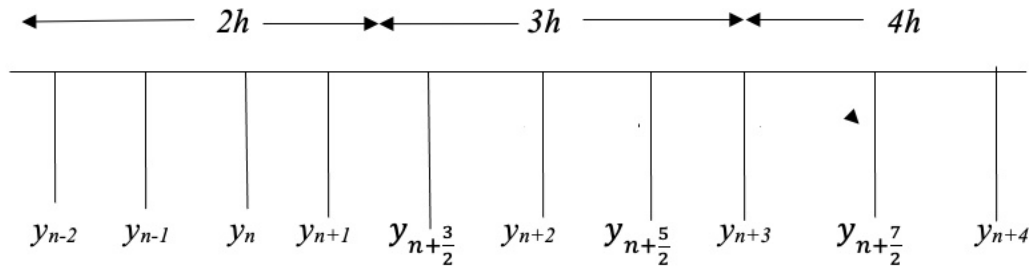


Figure 1: Block method with off-step points of order eight (8).

The step size h of the computed block is $3h$ with two off-step points, $y_{n+\frac{2}{3}}$ and $y_{n+\frac{5}{2}}$ as shown in figure 2.5 above. For our 3- point block method, the interval $[a, b]$ is divided into a series of blocks. For our research, the points y_{n-2}, y_{n-1} and y_n in the previous block are used to compute the solution of equation (2.1) at $x_n + 1, x_n + 2$ and $x_n + 3$ simultaneously. For our methodology, we consider the general k - step linear multistep method (LMM) of the form

$$\sum_{j=0}^k \alpha_j y_{n+j} = h \sum_{j=0}^k \beta_j f_{n+j} \quad (2.2)$$

In order to construct the new method, the linear multistep method in (2.2) is extended by using the off-step points. Therefore, the extension of the linear multistep method which involves $f(x, y)$ evaluated at an off-step point (x_{n+r}, y_{n+r}) , $0 < r < k$, and r not in $[0, 1, 2, \dots, k]$ takes the general form

$$\sum_{j=0}^{k-4} \alpha_j i y_{n+j-2} + \sum_{j=0}^{k-1} \alpha_{j+2} i y_{n+j-1/2} = h \beta_i f_{n+i} \quad (2.3)$$

Equation (2.3) is expanded to generate the block method. Hence, the approximation at the point $x_n + 1$ is given as follows:

$$\alpha_0, iy_{n-2} + \alpha_1, iy_{n-1} + \alpha_2, iy_n + \alpha_3, iy_{n+1} + \alpha_4, iy_{n+\frac{3}{2}} + \alpha_5, iy_{n+2} + \alpha_6, iy_{n+\frac{5}{2}} + \alpha_7, iy_{n+3} = h\beta_i f_{n+i} \quad (2.4)$$

Now, we obtain the first formula for $y_{n+\frac{3}{2}}$ by substituting $i = \frac{3}{2}$ in equation (2.4) to get

$$\alpha_0, \frac{3}{2}y_{n-2} + \alpha_1, \frac{3}{2}y_{n-1} + \alpha_2, \frac{3}{2}y_n + \alpha_3, \frac{3}{2}y_{n+1} + \alpha_4, \frac{3}{2}y_{n+\frac{3}{2}} + \alpha_5, \frac{3}{2}y_{n+2} + \alpha_6, \frac{3}{2}y_{n+\frac{5}{2}} + \alpha_7, \frac{3}{2}y_{n+3} = h\beta_{\frac{3}{2}} f_{n+\frac{3}{2}}. \quad (2.5)$$

We let $\alpha_4, \frac{3}{2} = 1$ by normalization that is to remove the arbitrariness of coefficients, so we have

$$\alpha_0, \frac{3}{2}y_{n-2} + \alpha_1, \frac{3}{2}y_{n-1} + \alpha_2, \frac{3}{2}y_n + \alpha_3, \frac{3}{2}y_{n+1} + y_{n+\frac{3}{2}} + \alpha_5, \frac{3}{2}y_{n+2} + \alpha_6, \frac{3}{2}y_{n+\frac{5}{2}} + \alpha_7, \frac{3}{2}y_{n+3} = h\beta_{\frac{3}{2}} f_{n+\frac{3}{2}} \quad (2.6)$$

Solving and expanding using the Taylor's series expansion, we get the following schemes

$$y_{n+1} = \frac{10584}{137975}f_{n+6}h - \frac{77364}{137975}f_{n+1}h - \frac{2852864}{1517725}y_{n+\frac{11}{2}} - \frac{1714176}{137975}y_{n+\frac{9}{2}} + \frac{1027971}{137975}y_{n+5} + \frac{254898}{27595}y_{n+4} - \frac{527436}{137975}y_{n+3} + \frac{344736}{137975}y_{n+2} - \frac{90846}{1517725}y_n:$$

$$y_{n+2} = -\frac{1225}{40514}f_{n+6}h - \frac{32235}{40514}f_{n+2}h + \frac{517120}{668481}y_{n+\frac{11}{2}} + \frac{7168}{1293}y_{n+\frac{9}{2}} - \frac{64190}{20257}y_{n+5} - \frac{89425}{20257}y_{n+4} + \frac{147980}{60771}y_{n+3} - \frac{10850}{60771}y_{n+1} + \frac{6370}{668481}y_n:$$

$$y_{n+3} = \frac{25}{1604}f_{n+6}h - \frac{1535}{1604}f_{n+3}h - \frac{13248}{30877}y_{n+\frac{11}{2}} - \frac{13120}{3609}y_{n+\frac{9}{2}} + \frac{12015}{6416}y_{n+5} + \frac{22275}{6416}y_{n+4} - \frac{1755}{5614}y_{n+2} + \frac{225}{6416}y_{n+1} - \frac{1585}{635184}y_n:$$

$$y_{n+4} = -\frac{24}{7495}f_{n+6}h - \frac{3684}{7495}f_{n+4}h + \frac{126976}{1236675}y_{n+\frac{11}{2}} + \frac{405504}{262325}y_{n+\frac{9}{2}} - \frac{19944}{37475}y_{n+5} - \frac{5072}{37475}y_{n+3} + \frac{828}{37475}y_{n+2} - \frac{104}{31479}y_{n+1} + \frac{109}{412225}y_n:$$

$$y_{n+\frac{9}{2}} = -\frac{96705}{503}f_{n+\frac{9}{2}}h + \frac{33075}{32192}f_{n+6}h - \frac{214515}{5533}y_{n+\frac{11}{2}} + \frac{17086545}{64384}y_{n+5} - \frac{31454325}{128768}y_{n+4} + \frac{721035}{32192}y_{n+3} - \frac{141183}{32192}y_{n+2} + \frac{45225}{64384}y_{n+1} - \frac{82565}{1416448}y_n:$$

$$y_{n+5} = \frac{200}{27239}f_{n+6}h + \frac{18420}{27239}f_{n+5}h - \frac{2529280}{6292209}y_{n+\frac{11}{2}} + \frac{1116160}{572019}y_{n+\frac{9}{2}} - \frac{16850}{27239}y_{n+4} + \frac{6940}{81717}y_{n+3} - \frac{3520}{190673}y_{n+2} + \frac{1775}{572019}y_{n+1} - \frac{238}{898887}y_n:$$

$$y_{n+\frac{11}{2}} = \frac{1063755}{3763757} f_{n+\frac{11}{2}} h - \frac{4002075}{240880448} f_{n+6} h - \frac{4916835}{3763757} y_{n+\frac{9}{2}} + \frac{881256915}{481760896} y_{n+5} + \frac{529607925}{963521792} y_{n+4} \\ - \frac{21717927}{240880448} y_{n+3} + \frac{5047515}{240880448} y_{n+2} - \frac{1757525}{481760896} y_{n+1} - \frac{306495}{963521792} y_n :$$

$$y_{n+6} = \frac{60}{307} h f_{n+6} - \frac{10}{10131} y_n + \frac{24}{2149} y_{n+1} - \frac{135}{2149} y_{n+2} + \frac{80}{307} y_{n+3} - \frac{450}{307} y_{n+4} \\ - \frac{1080}{307} y_{n+5} + \frac{20480}{6447} y_{n+\frac{9}{2}} + \frac{61440}{23639} y_{n+\frac{11}{2}} : \quad (2.7)$$

Collecting the coefficients of all $y_n, y_{n+1}, y_{n+2}, \dots$ in equation (2.7) and putting them in matrix form we have

$$\begin{pmatrix} \frac{90846}{1517725} & 1 & \frac{-344736}{137975} & \frac{527436}{137975} & \frac{-254898}{27595} & \frac{1714176}{137975} & \frac{-1027971}{137975} & \frac{2852864}{1517725} & 0 \\ \frac{-6370}{668481} & \frac{10850}{60771} & 1 & \frac{-147980}{60771} & \frac{89425}{20257} & \frac{-7168}{1293} & \frac{64190}{20257} & \frac{-517120}{668481} & 0 \\ \frac{1585}{635184} & \frac{-225}{6416} & \frac{1755}{5614} & 1 & \frac{-22275}{6416} & \frac{13120}{3609} & \frac{-12015}{6416} & \frac{13248}{30877} & 0 \\ \frac{-109}{412225} & \frac{104}{31479} & \frac{-828}{37475} & \frac{5072}{37475} & 1 & \frac{-405504}{262325} & \frac{19944}{37475} & \frac{-126976}{1236675} & 0 \\ \frac{82565}{1416448} & \frac{-45225}{64384} & \frac{141183}{32192} & \frac{-721035}{32192} & \frac{31454325}{128768} & 1 & \frac{-17086545}{64384} & \frac{214515}{5533} & 0 \\ \frac{238}{898887} & \frac{-1775}{572019} & \frac{3520}{190673} & \frac{-6940}{81717} & \frac{16850}{27239} & \frac{-1116160}{572019} & 1 & \frac{2529280}{6292209} & 0 \\ \frac{-306495}{963521792} & \frac{1757525}{481760896} & \frac{-5047515}{240880448} & \frac{21717927}{240880448} & \frac{-529607925}{963521792} & \frac{4916835}{3763757} & \frac{881256915}{481760896} & 1 & 0 \\ \frac{10}{10131} & \frac{-24}{2149} & \frac{135}{2149} & \frac{-80}{307} & \frac{450}{307} & \frac{-20480}{6447} & \frac{1080}{307} & \frac{-61440}{23639} & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+\frac{9}{2}} \\ y_{n+5} \\ y_{n+\frac{11}{2}} \\ y_{n+6} \end{pmatrix} \quad (2.8)$$

Also collecting the coefficients of $f_{n+1}, f_{n+2}, \dots$ in equation (2.7) and putting them in matrix form we have

$$\begin{pmatrix} \frac{-77364}{137975} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{10584}{137975} \\ 0 & \frac{-32235}{40514} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1225}{40514} \\ 0 & 0 & \frac{-1535}{1604} & 0 & 0 & 0 & 0 & 0 & \frac{25}{1604} \\ 0 & 0 & 0 & \frac{-3684}{7495} & 0 & 0 & 0 & 0 & \frac{-24}{7495} \\ 0 & 0 & 0 & 0 & \frac{-96705}{503} & 0 & 0 & 0 & \frac{33075}{32192} \\ 0 & 0 & 0 & 0 & 0 & \frac{18420}{27239} & 0 & 0 & \frac{200}{27239} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1063755}{3763757} & \frac{-4002075}{240880448} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{60}{307} \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+\frac{9}{2}} \\ f_{n+5} \\ f_{n+\frac{11}{2}} \\ f_{n+6} \end{pmatrix} h \quad (2.9)$$

To solve our Error Constant we let $y_n, y_{n+1}, y_{n+2}, \dots$ in equation (2.7) to be $\alpha_0, \alpha_1, \alpha_2, \dots$, and $f_{n+1}, f_{n+2}, \dots$, to be $\beta_0, \beta_1, \beta_2, \dots$ as follows



$$\alpha_0 = \begin{bmatrix} \frac{-90846}{1517725} \\ \frac{-6370}{668481} \\ \frac{1585}{635184} \\ \frac{-109}{412225} \\ \frac{82565}{1416448} \\ \frac{238}{898887} \\ \frac{-306495}{963521792} \\ \frac{10}{10131} \end{bmatrix}, \quad \alpha_1 = \begin{bmatrix} 1 \\ \frac{10850}{60771} \\ \frac{-225}{6416} \\ \frac{104}{31479} \\ \frac{-45225}{64384} \\ \frac{-1775}{572019} \\ \frac{1757525}{481760896} \\ \frac{-24}{2149} \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} \frac{-344736}{137975} \\ 1 \\ \frac{1755}{5614} \\ \frac{-828}{37475} \\ \frac{141183}{32192} \\ \frac{3520}{190673} \\ \frac{-5047515}{240880448} \\ \frac{135}{2149} \end{bmatrix}, \quad \alpha_3 = \begin{bmatrix} \frac{527436}{137975} \\ \frac{-147980}{60771} \\ 1 \\ \frac{5072}{37475} \\ \frac{-721035}{32192} \\ \frac{-6940}{81717} \\ \frac{21717927}{240880448} \\ \frac{-80}{307} \end{bmatrix},$$

$$\alpha_4 = \begin{bmatrix} \frac{-254898}{27595} \\ \frac{89425}{20257} \\ \frac{-22275}{6416} \\ 1 \\ \frac{31454325}{128768} \\ \frac{16850}{27239} \\ \frac{21717927}{240880448} \\ \frac{450}{307} \end{bmatrix}, \quad \alpha_5 = \begin{bmatrix} \frac{1714176}{137975} \\ \frac{-7168}{1293} \\ \frac{13120}{3609} \\ \frac{-405504}{262325} \\ 1 \\ \frac{-1116160}{572019} \\ \frac{4916835}{3763757} \\ \frac{-20480}{6447} \end{bmatrix}, \quad \alpha_6 = \begin{bmatrix} \frac{-1027971}{137975} \\ \frac{64190}{20257} \\ \frac{-12015}{6416} \\ \frac{19944}{37475} \\ \frac{-17086545}{64384} \\ 1 \\ \frac{-881256915}{481760896} \\ \frac{1080}{307} \end{bmatrix}, \quad \alpha_7 = \begin{bmatrix} \frac{2852864}{1517725} \\ \frac{-517120}{668481} \\ \frac{13248}{30877} \\ \frac{-126976}{1236675} \\ \frac{214515}{5533} \\ \frac{2529280}{6292209} \\ 1 \\ \frac{-61440}{23639} \end{bmatrix},$$

$$\alpha_8 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\beta_0 = \begin{bmatrix} \frac{-77364}{137975} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \beta_1 = \begin{bmatrix} 0 \\ \frac{-32235}{40514} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \beta_2 = \begin{bmatrix} 0 \\ 0 \\ \frac{-1535}{1604} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \beta_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{-3684}{7495} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \beta_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{33075}{32192} \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\beta_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{18420}{27239} \\ 0 \\ 0 \end{bmatrix}, \beta_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1063755}{3763757} \\ 0 \end{bmatrix}, \beta_7 = \begin{bmatrix} \frac{10584}{137975} \\ \frac{-1225}{40514} \\ \frac{25}{1604} \\ \frac{-24}{7495} \\ \frac{33075}{32192} \\ \frac{200}{27239} \\ \frac{-4002075}{240880448} \\ \frac{60}{307} \end{bmatrix}.$$

2.3 Error constants of the method

To get our error of the method in equation (2.6), we define the corresponding linear multistep method in equation (2.3) with Taylor's series expansion to get equation (2.10) below.

$$\begin{aligned}
 c_0 &= \sum_{j=0}^{15} \alpha_j = 0 \\
 c_1 &= \sum_{j=0}^{15} (j\alpha_j) - 2 \sum_{j=0}^{15} \beta_j = 0 \\
 c_2 &= \sum_{j=0}^{15} \frac{(j^2\alpha_j)}{2!} - 2 \sum_{j=0}^{15} j(\beta_j) = 0 \\
 &\vdots \\
 c_9 &= \sum_{j=0}^{15} \frac{(j^9\alpha_j)}{9!} - 2 \sum_{j=0}^{15} \frac{(j^8\beta_j)}{8!} = \begin{pmatrix} \frac{6699}{2207600} \\ \frac{5635}{5834016} \\ \frac{545}{1437184} \\ \frac{337}{6295800} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (2.10)
 \end{aligned}$$

This yields the error constant.

$$c_9 = \begin{pmatrix} \frac{6699}{2207600} \\ \frac{5635}{5834016} \\ \frac{545}{1437184} \\ \frac{337}{6295800} \end{pmatrix}.$$

This shows that, the method is of order eight (8) with the error constant as displayed in $c_9 = c_{p+1} \neq 0$. Hence, its proved that the method is consistent, since the order of the method, $p = 8 \geq 1$.

2.4 Zero - Stability

For the method to be convergent, it has to be consistent and zero-stable. The stability of the method is determined by the linear test equation as giving below

$$y' = \lambda y, \quad (2.11)$$

where λ is a complex constant with $Re(\lambda) < 0$.

Substituting equation (2.11) into equation (2.7) we have

$$\begin{aligned}
 y_{n+1} &= \frac{10584}{137975} h \lambda y_{n+6} - \frac{77364}{137975} h \lambda y_{n+1} - \frac{2852864}{1517725} y_{n+\frac{11}{2}} - \frac{1714176}{137975} y_{n+\frac{9}{2}} + \frac{1027971}{137975} y_{n+5} \\
 &+ \frac{254898}{27595} y_{n+4} - \frac{527436}{137975} y_{n+3} + \frac{344736}{137975} y_{n+2} - \frac{90846}{1517725} y_n.
 \end{aligned}$$

$$\begin{aligned}
 y_{n+2} &= -\frac{1225}{40514}h\lambda y_{n+6} - \frac{32235}{40514}h\lambda y_{n+2} + \frac{517120}{668481}y_{n+\frac{11}{2}} + \frac{7168}{1293}y_{n+\frac{9}{2}} - \frac{64190}{20257}y_{n+5} \\
 &\quad - \frac{89425}{20257}y_{n+4} + \frac{147980}{60771}y_{n+3} - \frac{10850}{60771}y_{n+1} + \frac{6370}{668481}y_n; \\
 y_{n+3} &= \frac{25}{1604}h\lambda y_{n+6} - \frac{1535}{1604}h\lambda y_{n+3} - \frac{13248}{30877}y_{n+\frac{11}{2}} - \frac{13120}{3609}y_{n+\frac{9}{2}} + \frac{12015}{6416}y_{n+5} \\
 &\quad + \frac{22275}{6416}y_{n+4} - \frac{1755}{5614}y_{n+2} + \frac{225}{6416}y_{n+1} - \frac{1585}{635184}y_n; \\
 y_{n+4} &= -\frac{24}{7495}h\lambda y_{n+6} - \frac{3684}{7495}h\lambda y_{n+4} + \frac{126976}{1236675}y_{n+\frac{11}{2}} + \frac{405504}{262325}y_{n+\frac{9}{2}} - \frac{19944}{37475}y_{n+5} \\
 &\quad - \frac{5072}{37475}y_{n+3} + \frac{828}{37475}y_{n+2} - \frac{104}{31479}y_{n+1} + \frac{109}{412225}y_n; \\
 y_{n+\frac{9}{2}} &= -\frac{96705}{503}h\lambda y_{n+\frac{9}{2}} + \frac{33075}{32192}h\lambda y_{n+6} - \frac{214515}{5533}y_{n+\frac{11}{2}} + \frac{17086545}{64384}y_{n+5} \\
 &\quad - \frac{31454325}{128768}y_{n+4} + \frac{721035}{32192}y_{n+3} - \frac{141183}{32192}y_{n+2} + \frac{45225}{64384}y_{n+1} - \frac{82565}{1416448}y_n; \\
 y_{n+5} &= \frac{200}{27239}h\lambda y_{n+6} + \frac{18420}{27239}h\lambda y_{n+5} - \frac{2529280}{6292209}y_{n+\frac{11}{2}} + \frac{1116160}{572019}y_{n+\frac{9}{2}} \\
 &\quad - \frac{16850}{27239}y_{n+4} + \frac{6940}{81717}y_{n+3} - \frac{3520}{190673}y_{n+2} + \frac{1775}{572019}y_{n+1} - \frac{238}{898887}y_n; \\
 y_{n+\frac{11}{2}} &= \frac{1063755}{3763757}h\lambda y_{n+\frac{11}{2}} - \frac{4002075}{240880448}h\lambda y_{n+6} - \frac{4916835}{3763757}y_{n+\frac{9}{2}} + \frac{881256915}{481760896}y_{n+5} \\
 &\quad + \frac{529607925}{963521792}y_{n+4} - \frac{21717927}{240880448}y_{n+3} + \frac{5047515}{240880448}y_{n+2} - \frac{1757525}{481760896}y_{n+1} - \frac{306495}{963521792}y_n; \\
 y_{n+6} &= \frac{60}{307}h\lambda y_{n+6} - \frac{10}{10131}y_n + \frac{24}{2149}y_{n+1} - \frac{135}{2149}y_{n+2} + \frac{80}{307}y_{n+3} - \frac{450}{307}y_{n+4} \\
 &\quad - \frac{1080}{307}y_{n+5} + \frac{20480}{6447}y_{n+\frac{9}{2}} + \frac{61440}{23639}y_{n+\frac{11}{2}}; \tag{2.12}
 \end{aligned}$$

Equation (2.12) can be written in matrix form as follows

Putting $H = h\lambda$, the stability polynomial $R(t, H)$ associated with our method is determined by solving the first characteristics polynomial $\det (At^2 - Bt) = 0$.

Therefore, the zero stability is determined by putting $H = h\lambda = 0$ which gives

$$t_1 = t_2 = t_3 = t_4 = t_5 = 0, \quad t_6 = t_7 = 1.$$

Since all the roots lie within $t \leq 1$, we conclude that our method is zero stable.

2.5 The A –Stability of the Method

A numerical method is said to be A- stable, if its region of absolute stability contains whole left plane. Therefore, since the region of absolute stability of our method contains whole left plane, we conclude that our method is A- stable. Below is the graph.

The stability region of the eight order Block Backward Differentiation formula with two off step points (BBDFO(8)). From the definition of A- stable which says that, A numerical method is said to

$$\begin{pmatrix}
 (1 + \frac{77364}{137975} h\lambda) - \frac{344736}{137975} \frac{527436}{137975} - \frac{254898}{27595} \frac{1714176}{137975} - \frac{1027971}{137975} \frac{2852864}{1517725} - \frac{10584}{137975} h\lambda & y_{n+1} \\
 \frac{10850}{60771} (1 + \frac{32235}{40514} h\lambda) - \frac{147980}{60771} \frac{89425}{20257} - \frac{7168}{1293} \frac{64190}{20257} - \frac{517120}{668481} \frac{1225}{40514} h\lambda & y_{n+2} \\
 -\frac{225}{6416} \frac{1755}{5614} (1 + \frac{1535}{1604} h\lambda) - \frac{22275}{6416} \frac{13120}{3609} - \frac{12015}{6416} \frac{13248}{30877} - \frac{25}{1604} h\lambda & y_{n+3} \\
 \frac{104}{31479} - \frac{828}{37475} \frac{5072}{37475} (1 + \frac{3684}{7495} h\lambda) - \frac{405504}{262325} \frac{19944}{37475} - \frac{126976}{1236675} \frac{24}{7495} h\lambda & y_{n+4} \\
 -\frac{45225}{64384} \frac{141183}{32192} - \frac{721035}{32192} \frac{31454325}{128768} (1 + \frac{96705}{503} h\lambda) - \frac{17086545}{64384} \frac{214515}{5533} - \frac{33075}{32192} h\lambda & y_{n+\frac{9}{2}} \\
 -\frac{1775}{572019} \frac{3520}{190673} - \frac{6940}{81717} \frac{16850}{27239} - \frac{1116160}{572019} (1 - \frac{18420}{27239} h\lambda) \frac{2529280}{6292209} - \frac{200}{27239} h\lambda & y_{n+5} \\
 \frac{1757525}{481760896} - \frac{5047515}{240880448} \frac{21717927}{240880448} - \frac{529607925}{963521792} \frac{4916835}{3763757} - \frac{881256915}{481760896} (1 - \frac{1063755}{3763757} h\lambda) & y_{n+\frac{11}{2}} \\
 -\frac{24}{2149} \frac{135}{2149} - \frac{80}{307} \frac{450}{307} - \frac{20480}{6447} \frac{1080}{307} - \frac{61440}{23639} (1 - \frac{60}{307} h\lambda) & y_{n+6}
 \end{pmatrix}$$

be A-stable if its region of absolute stability contains whole left plane, we then conclude that the BBDO(8) method is A- stable since its region of absolute stability contains whole left plane.

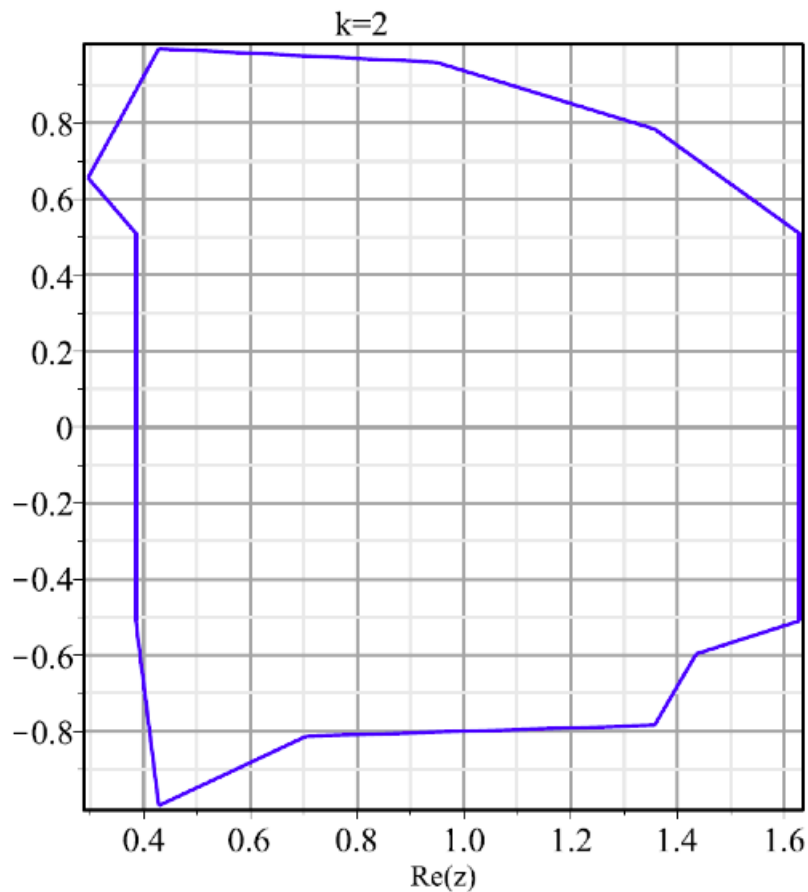


Figure 2: The Region of absolute stability when $k = 2$ with two off-step points.

3 Numerical results

Problem 1

$$y' = -1000(y - 1), \quad y(0) = 2, \quad x \in [0, 10].$$

Exact solution:

$$y(x) = e^{-1000x} + 1.$$

Problem 2:

$$y' = -\frac{y^3}{2}, \quad y(0) = 1, \quad x \in [0, 4].$$

Exact solution:

$$y(x) = \frac{1}{\sqrt{1+x}}.$$

Table 1: Numerical solution for problem 1 ($h = 0.1$)

x	Exact Solution	Numerical Solution	Error in order 8
1.0	1.0000	1.0000000070975453656	2.20×10^{-11}
2.0	1.0000	1.0000000000000000504	1.595×10^{-18}
3.0	1.0000	1.0000000000000000001	2.070×10^{-23}
4.0	1.0000	1.0000000000000000001	6.767×10^{-34}
5.0	1.0000	1.0000000000000000001	4.897×10^{-41}
6.0	1.0000	1.0000000000000000001	9.423×10^{-46}
7.0	1.0000	1.0000000000000000001	2.07×10^{-56}
8.0	1.0000	1.0000000000000000001	1.503×10^{-63}
9.0	1.0000	1.0000000000000000001	2.982×10^{-68}
10	1.0000	1.0000000000000000001	6.377×10^{-79}

Table 2: Numerical solution for problem 1 ($h = 0.1$)

h	MAX ERROR in order 8
10^{-3}	5.436×10^{-4}
10^{-4}	3.140×10^{-12}

Table 3: Numerical solution for problem 2 ($h = 0.1$)

x	Exact Solution	Numerical Solution	Error in order 8
0.4	0.845154254728519	0.845154576910226	3.222×10^{-7}
0.8	0.745355992499932	0.745356226209529	2.337×10^{-7}
1.2	0.674199862463243	0.674200035454122	1.730×10^{-7}
1.6	0.620173672946042	0.6201738085053073	1.356×10^{-7}
2.0	0.577350269189625	0.5773503787017336	1.095×10^{-7}
2.4	0.542326144546640	0.5423262353134918	9.077×10^{-8}
2.8	0.512989176042578	0.51298925288687785	7.684×10^{-8}
3.2	0.487950036474267	0.48795010261314547	6.614×10^{-8}
3.6	0.466252404120157	0.46625246182259146	5.770×10^{-8}
4.0	0.447213595499958	0.44721364642034503	5.092×10^{-8}

Table 4: Numerical solution for problem 2

h	MAX ERROR in order 8
10^{-3}	1.117×10^{22}
10^{-4}	1.119×10^{-30}

3.1 Discussion

The analysis of the proposed method shows that the method is of order eight (8). On implementing the method in maple environment, shows that the method is efficient for solving stiff differential equations. Also the tables of the results for problem 1 and 2 shows that the method is good and efficient.

3.2 Conclusion

A block backward formula for solving stiff initial value problems has been developed using backward differentiation method with two off-step points. The numerical solution of our method shows that the method is fast and efficient for solving stiff initial values problems. The stability analysis has also proven that the method is consistent and A-stable, hence convergent

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