International Journal of Mathematical Sciences and Optimization: Theory and Applications
11(3), 2025, Pages 68 - 79
https://doi.org/10.5281/zenodo.17584937

On Enriched Weakly Contractive Maps in Hilbert Spaces

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Article Info
Received: 06 March 2024 Revised: 15 December 2024
Accepted: 23 June 2025 Available online: 30 September 2025

Abstract

This paper introduces a class of enriched weakly contractive mappings for approximating a non-Picard average operator on a closed convex subset of Hilbert spaces. By imposing the enriched weakly mapping on the average operator, we establish and prove some results concerning convergence theorems (strong and weak), stability, and convergent rate. The validity and generality of the new class of enriched weakly mappings are examined with the aid of practical examples. The results harmonize and improve some recent results on enriched contractive-type mappings.

Keywords: Enriched weakly contractive mapping, Strong and weak convergence, Stability, Hilbert spaces.
MSC2010: 47H09, 47H10, 47H20.

1 Introduction

Let $(E, \|\cdot\|)$ be a normed vector space and T be a self-map of E. Also, let $F(T) = \{p \in X : p = Tp\}$ be the set of all fixed points of T. The map T is called a Banach contraction [1] if there exists a non-negative constant $a < 1$ such that

$$\|Tx - Ty\| \leq a \|x - y\|, \; \forall \; x, y \in E. \tag{1.1}$$

The Banach's contraction map (1.1) is a fundamental map that connects a broader family of contractive-type mappings in the literature and it often serves as a starting point for generalizations in the metrical fixed point theory. For instance, if $a = 1$, the condition

$$\|Tx - Ty\| \leq \|x - y\|, \; \forall \; x, y \in E. \tag{1.2}$$

defines a nonexpansive mapping. Also, by letting $a = a(s)$ be a function $a : [0, \infty) \to [0, 1)$ such that

$$\|Tx - Ty\| \leq a(\|x - y\|) \|x - y\|, \; \forall \; x, y \in E. \tag{1.3}$$

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Three significant independent results could be obtained from (1.3). That is, the map T satisfying (1.3) has a unique fixed point if (i) $a_d(s)$ is a decreasing function, see [2]; (ii) $a_g(s)$ is such that $a_g(s_n) \to 1$ (a is not continuous at 1) implies that $s_n \to 0$, see [3]; and (iii) $a_i(s)$ is an increasing function, see [4]. The above results are unified by the nonlinear contractive condition

$$\|Tx - Ty\| \leq \psi\left(\|x - y\|\right) \tag{1.4}$$

where $\psi : [0, \infty) \to [0, \infty)$ is a nondecreasing and upper semi-continuous function from the right. This is an equivalent condition in [5] if $\psi(s) < s$, for $s \in \mathbb{R}^+$.
In [6], the unification of both conditions (1.1) and (1.2), called a weakly contractive mapping, was introduced in a Hilbert space $\mathbb{H}$ setting as:

$$\|Tx - Ty\| \leq \|x - y\| - \phi\left(\|x - y\|\right), \ \forall\ x, y \in \mathbb{H}. \tag{1.5}$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and nondecreasing function with $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$. The result is presented as follows:

Theorem 1.1. *[6] If T is a weakly contractive map (1.5) on $\Omega \subset \mathbb{H}$, then it has a unique fixed point $p \in \Omega$.*

Several fixed point theorems exist in the literature regarding the existence and convergent properties of the weakly contractive map (1.5) for a sequence x_n defined by the Picard and non-Picard operators, see [7–17] and some references therein. Recently, some authors introduced the notion of enrichments to generalize the Banach contraction and nonexpansive maps as follows:

Definition 1.2. *[18] Let $(X, \|\cdot\|)$ be a linear normed space. A mapping $T : X \to X$ is said to be an enriched contraction if there exist $b \in [0, \infty)$ and $\theta \in [0, b + 1)$ such that*

$$\|b(x - y) + Tx - Ty\| \leq \theta \|x - y\|, \forall x, y \in X. \tag{1.6}$$

Definition 1.3. *[19] Let $(X, \|\cdot\|)$ be a linear normed space. A mapping $T : X \to X$ is said to be an enriched nonexpansive mapping if there exists $b \in [0, \infty)$ such that*

$$\|b(x - y) + Tx - Ty\| \leq (b + 1) \|x - y\|, \ \forall x, y \in X. \tag{1.7}$$

A unified and enriched condition of both (1.6) and (1.7) is presented as follows:

Definition 1.4. *Let $(X, \|\cdot\|)$ be a linear normed space. A mapping $T : X \to X$ will be called a unified enriched nonlinear contractive mapping if there exist $b \in [0, \infty)$, upper semi-continuous and nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\|b(x - y) + Tx - Ty\| \leq \psi_b\left(\|x - y\|\right), \ \forall x, y \in X. \tag{1.8}$$

For example, if $\psi_b(s) = a_g(s)s$, then a_g has the property that $a_g(s_n) \to b + 1$ (not continuous) as $s_n \to 0$, called an enriched Geraghty mapping.
Recent research papers have extended the notion of enrichments to study several contractive-type mappings for approximating non-Picard iterations in ambient spaces, see [20–28].
However, the present paper aims to introduce an enriched weakly contractive mapping to examine the existence and convergent properties of an average operator, called the Krasnoselskij operator [29]. Novelties of the paper that (i) it forms a class of the enriched Banach contraction mapping and other enriched contractive-type mappings. (ii) it mediates between two or more existing enriched conditions. (iii) it has a better convergent rate. (iv) it extends and generalizes the weakly contractive-type mappings in the literature.

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2 Enriched Weakly Contractive Mappings

We begin this section with the following definition:

Definition 2.1. *The set Φ^* is defined as the class of refined functions $\phi_{b,\theta} : \mathbb{R}^+ \to \mathbb{R}^+$ with the following property:*

ϕ_1: *$\phi_{b,\theta}$ is lower semi-continuous and non-decreasing function;*

ϕ_2: *$\phi_{b,\theta}(s) < (b+1)s$ for $b \in [0,\infty)$;*

ϕ_3: *$\phi_{b,\theta}(s) = 0$ if and only if $s = 0$; and*

ϕ_4: *$\phi_{b,\theta}$ is subadditive.*

For example, let $\phi_{b,\theta} : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $\phi_{b,\theta}(s) = \theta s$ for $\theta \in [0, b+1)$ and $s \in \mathbb{R}^+$, then $\phi_{b,\theta} \in \Phi^*$. Obviously, $\Phi^* = \Phi$ with $b = 0$. Thus, the class Φ is a subclass of Φ^*. We shall write ϕ_b to denote $\phi_{b,\theta}$ when there is no emphasis on θ. Other kinds of function ϕ_b include: $\phi_{b,\theta}(s) = \theta(e^s - 1)$; $\phi_{b,1}(s) = s^q$ for $q > 1$ etc.

Definition 2.2. *Let $(E, \|\cdot\|)$ be a linear normed space. A mapping $T : E \to E$ is said to be an enriched weakly contractive mapping if there exist $b \in [0,\infty)$ and $\phi_b \in \Phi^*$ such that*

$$\|Tx - Ty\| \leq (b+1)\|x - y\| - \phi_b(\|x - y\|), \ \forall\, x, y \in E. \tag{2.1}$$

Henceforth, we refer to T as a $\phi_{b,\theta}$-enriched weakly contractive mapping. Obviously, condition (2.1) is a weakly contractive mapping with $\phi_{b,\theta} = \phi_{0,a}(= \phi)$, that is, any weakly contractive map T is a $\phi_{0,a}$-enriched weakly contractive map for $a \in (0,1)$. The following are consequences of (2.1):

I. If $\phi_{b,0} \in \Phi^*$, then condition (2.1) becomes the enriched nonexpansive mapping.

II. If $\phi_{b,\theta} \in \Phi^*$ for $\theta \in (0, b+1)$, then condition (2.1) is a class of enriched Banach contraction mapping.

III. If $\phi_{b,b+1} \in \Phi^*$, then there is no discrepancy between (1.6) and (1.7).

Properties I. and II. are clear indication that the new enriched weakly contractive mapping harmonizes some classes of enriched contractive-type mappings. The following example shows that the enriched weakly contractive mapping strictly includes the weakly contractive mapping and the Banach's condition.

Example 2.3. *Let $E = [0,1]$ be equipped with the usual metric which induces a norm $\|\cdot\|$ and $T : E \to E$ be defined by $Tx = 1 - x$; $\forall\, x, y \in E$ with $F(T) = \{\frac{1}{2}\}$. Then,*

(i) T is not a Banach contraction mapping (1.1).

(ii) T is not a weakly contractive mapping (1.5).

(iii) T is an enriched weakly contractive mapping.

(i) Assume that the map T is a Banach contraction mapping (1.1) for $x, y \in E$, then

$$|Tx - Ty| = |x - y| \leq a\,|x - y|\,.$$

But

$$|x - y| \leq a\,|x - y| < |x - y|$$

is a contradiction.
(ii) Also, suppose that the map T is a weakly contractive mapping (1.5) for $x, y \in E$, we have

$$|Tx - Ty| = |x - y| \leq |x - y| - \phi(|x - y|).$$

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Therefore,

$$|x-y| \leq |x-y| - \phi(|x-y|) < |x-y|$$

is a contradiction.
(iii) Here, we show that T is an enriched weakly contractive mapping. Indeed,

$$|b(x-y)+Tx-Ty| = |1-b|\,|x-y| \leq (1+b)\,|x-y| - \phi_b(|x-y|)$$

where $\phi_b \in \Phi^*$. For $x \neq y$, the last inequality becomes

$$(b+1)\,|x-y| - 2\,|x-y| \leq (b+1)\,|x-y| - \phi_b(|x-y|).$$

Consequently,

$$\phi_b(|x-y|) \leq 2\,|x-y| < (b+1)\,|x-y|\,.$$

By the hypothesis on ϕ_b, T is an enriched weakly contractive mapping for $b > 1$. More so, by defining $\phi_b(s) = \theta s$ for $\theta \in [0, b+1)$, we obtain from the last inequality that $\theta = 2$. Thus, the inequality holds only if $b \in (1, \infty)$.

Example 2.4. *[19] Let $E = [\frac{1}{2}, 2]$ be endowed with the usual norm and $T : E \to E$ be defined by $Tx = \frac{1}{x}$, for all $x \in [\frac{1}{2}, 2]$ with $F(T) = \{1\}$. Then,*

(i) T is not a nonexpansive mapping;

(ii) T is not a quasi-nonexpansive mapping; and

(iii) T is a $\phi_{b,5}$-enriched weakly contractive mapping.

The justifications that (i) and (ii) hold were discussed in [19]. For (iii), the enriched weakly contractive mapping reduces, in this case, to

$$|b(x-y)+Tx-Ty| = |b - \frac{1}{xy}|\,|x-y| \leq (b+1)\,|x-y| - \phi_b(|x-y|),$$

But then $\left||b+1| - |1+\frac{1}{xy}|\right| \leq |b - \frac{1}{xy}|$ for all $x, y \in [\frac{1}{2}, 2]$. Moreover, for $b > 4$,

$$(b+1)\,|x-y| - (1+\frac{1}{xy})\,|x-y| \leq (b+1)\,|x-y| - \phi_b(|x-y|)$$

which leads to

$$\phi_b(|x-y|) \leq (1+\frac{1}{xy})\,|x-y|\,.$$

Thus, the last inequality is satisfied if only

$$(1+\frac{1}{xy})\,|x-y| < (b+1)\,|x-y|\,.$$

By letting $\theta = \max\{1+(xy)^{-1} : x, y \in [\frac{1}{2}, 2]\}$, then $\phi_b(s) = \theta s = 5s$ for all $s \in [0, \infty)$. Obviously, $\phi_b(s) \in \Phi^*$. Hence, T is a $\phi_{b,5}$-enriched weakly contractive mapping.

Remark 2.5. *Observe that the Picard operator fails to approximate the fixed point in Examples 2.3 and 2.4. Thus, to approximate fixed point of enriched weakly contractive mapping, we consider a more relaxing operator called the Krasnoselskij-averaged operator $T_\lambda = (1-\lambda)I + \lambda T$. In the sequel, we shall prove some results concerning existence properties, convergent (strong and weak), stability and convergent rate using the operator T_λ with imposition of condition (2.1).*

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https://doi.org/10.5281/zenodo.17584937

3 Main Results

The main results are presented as follows:

Theorem 3.1. *Let $\mathbb{H}$ be a Hilbert space and let $\mathcal{K}$ be a closed convex subset of $\mathbb{H}$. Suppose $T_\lambda : \mathcal{K} \to \mathcal{K}$ satisfies the enriched weakly contraction mapping, then T_λ has a unique fixed point.*

Proof: Let T_λ be an enriched weakly contractive mapping and let $\lambda = \frac{1}{b+1}$. Obviously, $\lambda \in (0,1)$ for all $b > 0$. Then,

$$\left\| (\frac{1}{\lambda} - 1)(x-y) + Tx - Ty \right\| \leq (b+1) \|x-y\| - \phi_b (\|x-y\|)$$

By applying λ to both sides, this gives

$$\|T_\lambda x - T_\lambda y\| \leq \|x-y\| - \phi(\|x-y\|)$$

where $\phi = \lambda \phi_b$. Since $\phi \in \Phi^*$, it follows that T_λ is a weakly contractive map. By Theorem 1.1, T_λ has a unique fixed point.

Theorem 3.2. *Let $T_\lambda : \mathcal{K} \to \mathcal{K}$ be an enriched weakly contraction mapping on $\mathcal{K} \subset \mathbb{H}$ with the set of fixed point $F(T) \neq \emptyset$. For $x_0 \in \mathcal{K}$, the sequence x_n given by*

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n$$

where $\lambda \in (0,1)$, converges to a unique fixed point in $F(T_\lambda)$.

Proof: Let $x_0 \in \mathcal{K}$ be given and $F(T) \subset \mathcal{K}$. By letting x_n be a sequence defined by

$$x_{n+1} = T_\lambda x_n, \ n = 0, 1, 2, \ldots$$

Then,

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|T_\lambda x_n - T_\lambda x_{n-1}\| \\ &\leq \|x_n - x_{n-1}\| - \lambda \phi_b (\|x_n - x_{n-1}\|) \end{aligned}$$

Let $\upsilon_n = \|x_n - x_{n-1}\|$, then by the property on ϕ_b, we have

$$\upsilon_{n+1} \leq \upsilon_n - \lambda \phi_b(\upsilon_n) \leq \upsilon_n \tag{3.1}$$

This means that the sequence $\{\upsilon_n\}$ is nonincreasing which converges to some nonnegative number υ, that is, for large n,

$$\upsilon = \liminf \upsilon_n = \liminf \upsilon_{n+1}$$

It follows from (3.1) that

$$\upsilon \leq \upsilon - \lambda \phi_b(\upsilon)$$

which further implies that $\phi_b(\upsilon) \leq 0$, and by hypothesis on ϕ, $\upsilon = 0$.
Next is to show that x_n is a Cauchy sequence. Suppose, otherwise, that x_n is not Cauchy, then for $\varepsilon > 0$, there exist positive integers m_k, n_k with $n_k \geq m_k > k$ such that

$$\|x_{n_k} - x_{m_k}\| \geq \varepsilon \text{ and } \|x_{n_k+1} - x_{m_k}\| < \varepsilon$$

Now by (3.1), we have

$$\begin{aligned} \varepsilon^2 \leq \|x_{n_k} - x_{m_k}\|^2 \leq \|x_{n_k+1} - x_{n_k}\|^2 - 2 < x_{n_k+1} - x_{n_k}, x_{n_k+1} - x_{m_k} > \\ + \|x_{n_k+1} - x_{m_k}\|^2 \ < \varepsilon^2 \end{aligned}$$

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This implies that

$$\|x_{n_k} - x_{m_k}\| = \|x_{n_k+1} - x_{m_k}\| = \varepsilon$$

Also,

$$\begin{aligned}\varepsilon = \left\|x_{n_{k+1}} - x_{m_k}\right\| &\leq \|T_\lambda x_{n_k} - T_\lambda x_{m_k}\| + \left\|x_{m_{k+1}} - x_{m_k}\right\| \\ &\leq \|x_{n_k} - x_{m_k}\| - \lambda\phi_{_b}\left(\|x_{n_k} - x_{m_k}\|\right) \\ &= \varepsilon - \lambda\phi_{_b}\left(\varepsilon\right)\end{aligned}$$

Consequently, $\phi_{_b}\left(\varepsilon\right) \leq 0$. By hypothesis, $\varepsilon = 0$ is a contradiction.
Therefore, the sequence x_n is Cauchy. By completeness of $\mathcal{K} \subset \mathbb{H}$, the sequence x_n converges (strongly) to $p \in \mathcal{K}$, that is,

$$\lim_{n\to\infty} < x_n - p, x_n - p >= \lim_{n\to\infty} \|x_n - p\|^2 \to 0$$

More so, by the continuity of T_λ, we have

$$\|T_\lambda x_n - T_\lambda p\|^2 \leq [\|x_n - p\| - \lambda\phi_{_b}\left(\|x_n - p\|\right)]^2 < \lim_{n\to\infty} \|x_n - p\|^2 \to 0$$

Next is to prove that p is fixed point of $T_\lambda p$, that is, (using the last inequality)

$$\|p - T_\lambda p\| \leq \|p - x_{n+1}\| + \|T_\lambda x_n - T_\lambda p\| \to 0$$

implies that $p = T_\lambda p$, as required.
Uniqueness: Suppose $p^* \neq p$, where p^* is another fixed point of T_λ. Then,

$$\begin{aligned}\|p^* - p\| &= \|T_\lambda p^* - T_\lambda p\| \\ &\leq \|p^* - p\| - \phi(\|p^* - p\|) < \|p^* - p\|\end{aligned}$$

is a contradiction. Hence, p is a unique fixed point of T_λ.

Corollary 3.3. *Let $T_\lambda \mathcal{K} \to \mathcal{K}$ be $(0;\phi_{_0})$-enriched weakly contractive mapping on $\mathcal{K} \subset \mathbb{H}$ with the set of fixed point $F(T_\lambda) \neq \emptyset$. For $x_0 \in \mathcal{K}$, the approximating sequence defined by $x_{n+1} = T_\lambda x_n$, $n = 0, 1, 2, \ldots$ converges strongly to a fixed point of T.*

The proof is immediate from Theorem 3.2, see also Theorem 3.1 in [6] for similar approach. The estimation of Theorem 3.2 is presented as follows:

Theorem 3.4. *Let $T_\lambda : \mathcal{K} \to \mathcal{K}$ be an enriched weakly contractive mapping on $\mathcal{K} \subset \mathbb{H}$ with the set of fixed point $F(T) \neq \emptyset$. Then, the following estimate holds:*

$$\|x_n - p\| \leq \Psi^{-1}\left[\Psi(\|x_0 - p\|) - \lambda n\right]$$

where $\Psi(s) = \int \frac{ds}{\psi(s)}$ and Ψ^{-1} is its inverse.

Proof: Since T_λ is an enriched weakly contraction mapping on $\mathcal{K}$ with $F(T) \neq \emptyset$. By Theorem 3.2 T has a fixed point p in $F(T)$. Select $x_0 \in \mathcal{K}$ and define $x_{n+1} = T_\lambda x_n$, then we have

$$\|x_{n+1} - p\| \leq \|x_n - p\| - \lambda\phi_{_b}\left(\|x_n - p\|\right)$$

Let $\varrho_n = \|x_n - p\|$ be the estimate at each n step so that

$$\lambda\phi_{_b}(\varrho_n) \leq \varrho_n - \varrho_{n+1}$$

This implies ϱ_n is a non-increasing and nonnegative sequence. Therefore, it converges to e, say, such that $\phi_{_b}(e) \leq 0$. By hypothesis, this implies that $e = 0$.
More so, if there exists $\varrho^* \in [\varrho_{n+1}, \varrho_n]$ such that $\phi_{_b}(\varrho_n) \geq \phi_{_b}(\varrho^*)$ for each n, then

$$\Psi(\varrho_n) - \Psi(\varrho_{n+1}) = \int_{\varrho_{n+1}}^{\varrho_n} \frac{dt}{\phi(t)} = \frac{\varrho_n - \varrho_{n+1}}{\phi(\varrho)} \geq \frac{\varrho_n - \varrho_{n+1}}{\phi_{_b}(\varrho_n)} \geq \lambda$$

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This further gives

$$\Psi(\varrho_{n+1}) \leq \Psi(\varrho_n) - \lambda \leq \Psi(\varrho_{n-1}) - 2\lambda \leq \Psi(\varrho_{n-2}) - 3\lambda \leq \cdots \leq \Psi(\varrho_0) - \lambda(n+1).$$

By transitivity, we have the estimate

$$\varrho_{n+1} \leq \Psi^{-1}\left[\Psi(\varrho_0) - \lambda(n+1)\right]$$

3.1 Strong and Weak Convergence Theorems

Here, we prove some convergence theorems (strong and weak) based on enriched weakly contractive mapping as a class of enriched nonexpansive mapping using the approach in [19,30]. We assume that an averaged mapping of the averaged mapping T is an averaged mapping, that is, for $\lambda_1, \lambda_2 \in (0,1)$, then $T_\lambda = (1-\lambda)I + \lambda T$, where $\lambda = \lambda_1\lambda_2$, see [19, 27] for details. The following Lemma is useful for the next theorem.

Lemma 3.5. *Let $\mathcal{K}$ be a bounded closed convex subset of a Hilbert space $\mathbb{H}$ and $T : \mathcal{K} \to \mathcal{K}$ be a nonexpansive and demicompact operator. Then, the set $F(T)$ of fixed points of T_λ is a nonempty convex set.*

Theorem 3.6. *Let $T_\lambda : \mathcal{K} \to \mathcal{K}$ be an enriched weakly contractive mapping and demicompact on $\mathcal{K} \subset \mathbb{H}$. Then,*

I. $F(T)$ is a nonempty convex set; and

II. for $x_0 \in \mathcal{K}$, the approximate sequence x_n given by

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n$$

where $\lambda \in (0,1)$, converges (strongly) to a fixed point of T.

Proof: I. Since $T_\lambda : \mathcal{K} \to \mathcal{K}$ is an enriched weakly contractive mapping, then it follows from Theorem 3.1 that T_λ is a nonexpansive mapping, that is,

$$\|T_\lambda x - T_\lambda y\| \leq \|x-y\| - \phi\left(\|x-y\|\right) \leq \|x-y\|$$

and its convexity is through Lemma 3.5.
II. Next is to show that the sequences $\{x_n - T_\lambda x_n\}$ converge strongly to zero. Now, let $x_0 \in \mathcal{K}$ and $p \in F(T) \subset \mathcal{K}$, then

$$\|x_{n+1} - p\|^2 \leq (1-\lambda)^2 \|x_n - p\|^2 + \lambda^2 \|Tx_n - Tp\|^2 + 2\lambda(1-\lambda) < x_n - p, Tx_n - Tp > \tag{3.2}$$

On the other hand, for positive constant μ such that $\mu^2 \leq \lambda(1-\lambda)$, we have

$$\mu^2 \|x_n - Tx_n\|^2 = \mu^2 \|x_n - p\|^2 + \mu^2 \|Tx_n - p\|^2 - 2\mu^2 < x_n - p, Tx_n - p > \tag{3.3}$$

By adding (3.2) and (3.3), this gives

$$\begin{aligned}\mu^2 \|x_n - T_\lambda x_n\|^2 + \|x_{n+1} - p\|^2 \leq{}& [\mu^2 + (1-\lambda)^2] \|x_n - p\|^2 + (\mu^2 + \lambda^2) \|Tx_n - Tp\|^2 \\ &+ 2[\lambda(1-\lambda) - \mu^2] \|x_n - p\| \|Tx_n - p\|\end{aligned}$$

where $< x_n - p, Tx_n - p > \leq \|x_n - p\| \|Tx_n - p\|$ (Cauchy-Schwarz inequality). Using $\mu^2 = \min_\lambda\{\lambda(1-\lambda)\}$ and condition of the theorem in the last inequality, we have

$$\mu^2 \|x_n - T_\lambda x_n\|^2 + \|x_{n+1} - p\|^2 \leq (1-\lambda) \|x_n - p\|^2 + \lambda \left[\|x_n - p\| - \lambda\phi_b\left(\|x_n - p\|\right)\right]^2$$

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HTTPS://DOI.ORG/10.5281/ZENODO.17584937

Also, since $\phi_b \in \Phi^*$, we have

$$\frac{\mu^2 \|x_n - T_\lambda x_n\|^2 + \|x_{n+1} - p\|^2}{\lambda} \leq (\frac{1}{\lambda} - 1) \|x_n - p\|^2 + \|x_n - p\|^2 = \frac{1}{\lambda} \|x_n - p\|^2$$

Therefore,

$$\|x_n - T_\lambda x_n\|^2 \leq \frac{1}{\mu^2} \left(\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \right) \tag{3.4}$$

Applying limit to (3.4) as $n \to \infty$ gives

$$\lim_{n\to\infty} \|x_n - T_\lambda x_n\|^2 \to 0$$

Hence, the sequence $\{x_n - T_\lambda x_n\}$ converges strongly to 0. By the demicompactness of T_λ, there exists a subsequence $\{x_{n_k}\}$ which is strongly convergent such that $x_{n_k} \to p$, where $p \in F(T_\lambda)$. Also, due to the nonexpansiveness (continuity) of T_λ, we have that $T_\lambda x_{n_k} \to T_\lambda p$ such that $T_\lambda p = p$. As required.

Remark 3.7. *We note the following remark from Theorem 3.6: (i) The class of enriched weakly contractive mapping is weaker than the class of enriched nonexpansive mapping; (ii) If the demicompactness assumption is dropped, then T_λ converges weakly to the fixed point. This is presented in the next theorem.*

Theorem 3.8. *Let $T : \mathcal{K} \to \mathcal{K}$ be an enriched weakly contractive mapping on $\mathcal{K} \subset \mathbb{H}$ with $F(T) \neq \emptyset$. Then, for $x_0 \in \mathcal{K}$, the approximate sequence x_n, given by $x_{n+1} = (1-\lambda)x_n + \lambda T x_n$, converges weakly to a fixed point of T_λ.*

Proof: Since T_λ is an enriched weakly contractive mapping, it follows from Theorem 3.1 that $F(T_\lambda) \neq \emptyset$. Now, we need to show that the subsequence x_{n_k} if converges weakly to a certain fixed point p_0, then p_0 is a fixed point of T_λ and of T, where $p_0 = p$. Suppose that $\{x_{n_k}\}$ does not converge weakly to (p, q), then by hypothesis of the theorem, we have

$$\begin{aligned} \|x_{n_k} - T_\lambda p_0\| &\leq \|T_\lambda x_{n_k} - T_\lambda p_0\| + \|x_{n_k} - T_\lambda x_{n_k}\| \\ &\leq \|x_{n_k} - p_0\| - \lambda \phi_b \left(\|x_{n_k} - p_0\| \right) + \|x_{n_k} - T_\lambda x_{n_k}\| \\ &\leq \|x_{n_k} - p_0\| + \|x_{n_k} - T_\lambda x_{n_k}\| \end{aligned} \tag{3.5}$$

Since $\|x_{n_k} - T_\lambda x_{n_k}\| \to 0$ as $n \to \infty$, by hypothesis of Theorem 3.6, then inequality (3.5) becomes

$$\|x_{n_k} - T_\lambda p_0\| - \|x_{n_k} - p_0\| \leq 0 \tag{3.6}$$

The rest of the proof is a routine of Theorem 5 in [19].

Remark 3.9. *A more general result is obtained when the condition $F(T_\lambda) \neq \emptyset$ is removed from Theorem 3.8 is the next theorem.*

Theorem 3.10. *Let $T : \mathcal{K} \to \mathcal{K}$ be an enriched weakly contractive mapping on $\mathcal{K} \subset \mathbb{H}$. Then, for $x_0 \in \mathcal{K}$, the approximate sequence x_n given by, $x_{n+1} = (1 - \lambda)x_n + \lambda T x_n$, converges weakly to a fixed point of T_λ.*

Proof: Let $F(T_\lambda) \subset \mathcal{K}$ be the set fixed points of T_λ. Since T_λ is an enriched weakly contractive mapping on $\mathcal{K}$, for $p \in \mathcal{K}$ and $x_{n+1} = T_\lambda x_n$, then

$$\|x_{n+1} - p\| \leq \|x_n - p\| - \phi_b \left(\|x_n - p\| \right) \leq \|x_n - p\|$$

Let $\chi(p) = \lim_{n\to\infty} \|x_n - p\|$, then χ is well-defined and is a lower semicontinuous convex function on $F(T_\lambda)$. The proof also has similar arguments as that of Theorem 6. in [19] and it is omitted.

IJMSO

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https://doi.org/10.5281/zenodo.17584937

3.2 Stability

Here, we discuss briefly the stability of enriched weakly contractive mapping. Let $\{y_n\}$ be an arbitrary sequence in $\mathcal{K}$ which is close enough to $\{x_n\} \subset \mathcal{K}$ and let γ_n be a nonnegative term at each n-step such that $x_n = y_n + \gamma_n$, for $n = 1, 2, \ldots$. Stability set in if the sequence x_n converges to a fixed point p, then y_n converges if and only if $\gamma_n \to 0$ as $n \to \infty$. Let x_n be defined by the approximate sequence $x_{n+1} = T_\lambda x_n$, the stability of T_λ belonging to the class enriched weakly contractive mapping is stated as follow:

Theorem 3.11. *Let $T : \mathcal{K} \to \mathcal{K}$ be an enriched weakly contractive mapping on $\mathcal{K} \subset \mathbb{H}$ with $F(T_\lambda) \neq \emptyset$. Then, for $x_0 \in \mathcal{K}$, the approximate sequence x_n given by, $x_{n+1} = (1-\lambda)x_n + \lambda T x_n$, is T_λ-stable.*

Proof: Let $\{y_n\} \subset K$ be an arbitrary sequence and let $\gamma_n = \|y_n - T_\lambda y_n\|$. Then,

$$\begin{aligned}
\|y_n - p\|^2 &= \|y_n - T_\lambda y_n - (p - T_\lambda y_n)\|^2 \\
&\leq \|y_n - T_\lambda y_n\|^2 - 2 < y_n - T_\lambda y_n, T_\lambda y_n - p > + \|T_\lambda y_n - p\|^2 \\
&\leq \gamma_n^2 - 2 < \gamma_n, T_\lambda y_n - p > + (\|y_n - p\| - \lambda \phi_b(\|y_n - p\|))^2
\end{aligned}$$

This implies that

$$\|y_n - p\|^2 \leq \pi_n + (\|y_n - p\| - \lambda \phi_b(\|y_n - p\|))^2$$

where $\pi_n = \gamma_n^2 - 2 < \gamma_n, T_\lambda y_n - p >$. By hypothesis of Theorem 3.6,

$$\lim_{n\to\infty} \gamma_n^2 = \lim_{n\to\infty} \|y_n - T_\lambda y_n\|^2 \Rightarrow \lim_{n\to\infty} \pi_n \to 0$$

Therefore,

$$\|y_n - p\|^2 \leq (\|y_n - p\| - \lambda \phi_b(\|y_n - p\|))^2$$

which is a similar routine in Theorem 3.6. Hence, y_n converges to $p \in F(T)$.
Conversely, assume $\|y_n - p\| \to 0$ for large n, following a similar task, we obtain $\gamma_n \to 0$ for large n. Therefore, T_λ is stable.

3.3 Practical Example

We illustrate the condition (2.1) further with the following example.

Example 3.12. *Let $T : X \to X$ be given by $Tx = \cos(2\sqrt{x})$, where $X = [0, 1]$ and fixed point $p = 0.36092$. Then, T is an enriched weakly contractive map.*

Indeed, for all $x, y \in [0, 1]$ with $x < y$ and $\cos(2\sqrt{x}) = 1 - 2x + \frac{2}{3}x^2 - \frac{4}{45}x^3 + \gamma_1$, where γ_1 represents all other terms, we have

$$\begin{aligned}
\|b(x-y) + Tx - Ty\| &= \left| b(x-y) + 1 - 2x + \frac{2}{3}x^2 + \gamma_1 - (1 - 2y + \frac{2}{3}y^2 + \gamma_2) \right| \\
&= \left| (2-b)(y-x) - \frac{2}{3}(y^2 - x^2) + \frac{4}{45}(y^3 - x^3) + \gamma \right| \\
&\leq (b+1)\,|x-y| - \phi_b\,(|x-y|) + \gamma
\end{aligned}$$

where $\gamma = \gamma_1 - \gamma_2$. Since $x < y$, then $(y-x)^q \geq (y-x)^{q+1}$ and $y^q - x^q \geq (y-x)^q$ hold for $q \in (1, \infty)$. Thus,

$$|2-b|\,|x-y| - \frac{26}{45}\,|x-y|^2 \leq (b+1)\,|x-y| - \phi_b\,(|x-y|)$$

which further gives

$$(b+1)\left|x-y\right|-\left[(2b-1)\left|x-y\right|+\frac{26}{45}\left|x-y\right|^2\right]\leq(b+1)\left|x-y\right|-\phi_{_b}\left(\left|x-y\right|\right)$$

By equating terms, we have that $\phi_{_b}(s)=\frac{26}{45}s^2+(2b-1)s$ for all $s\in\mathbb{R}^+$. Obviously, $\phi_{_b}(s)\in\Phi^*$ for all $b\in[0,1)$. Hence, all hypotheses are fulfilled. Now, by means of an iterative sequence $x_{n+1}=T_\lambda x_n$ using suitable $\lambda\in(0,1)$, the sequence x_n converges to $p=0.36092$ with the estimate (by means of Theorem 3.4),

$$|x_n-p|\leq\frac{1}{\left|x_{_0}-p\right|^{-1}+\frac{52}{135}n}$$

with $b=0.5$ and for any initial point $x_{_0}\in[0,1]$.

Remark 3.13. *(i) The weakly contractive map is not suitable for Example 3.12; (ii) the enriched contraction map is only suitable if* $b\in[0,1)$.

We end the section by presenting a subclass of enriched weakly contractive map.

Definition 3.14. *Let* $(E,\|\cdot\|)$ *be a linear normed space. A map* $T:E\to E$ *will be called an enriched weakly quasi-contractive map if there exist a fixed point* $p\in F(T)$, $b\in[0,\infty)$ *and* $\phi_{_b}\in\Phi^*$ *such that*

$$\|Tx-p\|\leq(b+1)\left\|x-p\right\|-\phi_{_b}\left(\left\|x-p\right\|\right),\ \forall\ x\in E. \tag{3.7}$$

Remark 3.15. *Both Examples 2.3 and 2.4 satisfy the enriched weakly quasi-contractive map. More so, the convergence theorems are valid by using the condition (3.7).*

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