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Numerical Solution of First Order Ordinary Differential Equations using Compact Finite Difference scheme

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Abstract

In this paper, a fourth order Compact Finite Difference Scheme(CFD) for the numerical solution of first order initial value problems (IVPs) of Ordinary Differential Equations (ODEs) is discussed. Compact finite difference scheme is a class of numerical methods that are particularly designed for solving Partial Differential Equations (PDEs). However, in this paper, we consider the compact finite difference scheme for approximating the numerical solution of ordinary differential equations. The application of this scheme enables the solution of first-order ODEs across all grid points in just one computational sweep (single iteration), rather than requiring repeated iterative updates. Numerical examples have been included to demonstrate the accuracy of the scheme and Numerical results compared with the exact solution and other existing methods from recent literature. The scheme is shown to be efficient for the numerical integration of first order differential equations.

Keywords: Initial Value Problem, Ordinary Differential Equation, Implementation, Compact Difference Scheme, Grid Point.
MSC2010: 65M06, 65N35, 35F50.

1 Introduction

In science and engineering, mathematical models are developed to analyze physical phenomena such as population growth, disease transmission, fluid flow, satellite tracking, and celestial mechanics (Atteh and Edogbanya [1]). Many of these models consist of ordinary differential equations, which are often unsolvable analytically (Imoni et al. [2]). Consequently, numerical methods are employed to obtain approximate solutions (Debnath [3]).

In this paper, we consider the approximate solution of first-order IVPs of ODEs of the form:

$$y^{'} = f(x, y), y(x_0) = y_0, x \in [x_0, x_n] \tag{1.1}$$

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Many numerical methods have been derived and implemented for solving (1.1). The earliest among them are the Euler and Runge-Kutta methods, introduced by Euler [4], Runge [5], and Kutta [6]. These were followed by the finite difference and finite element methods, as presented by Hrennikoff [7] and Thom and Apelt [8]. Later, Linear Multistep Methods were developed using interpolation and collocation techniques and applied in block form, as seen in the works of Atteh and Edogbanya [1], Imoni et al. [2], Butcher [9], Sirisena et al. [10], and Rufai et al. [11].

Several authors, including Simos [12], Simos and Aguiar [13], and Ochoche [14], modified the Euler and Runge-Kutta methods to develop new numerical schemes for solving (1.1). Similarly, Chung et al. [15] proposed modifications to the finite element method for the same purpose.

Most of these methods require multiple iterations to compute results across all grid points. However, in this research, we propose the Compact Finite Difference Scheme combined with a one-sided scheme to approximate (1.1). This approach yields results for all grid points in a single iteration, enhancing computational efficiency.

2 Derivation of the Schemes

Let us assume that every discritized real valued function $y(x)$ and its first derivative satisfies the equation below:

$$\alpha y'_{i-1} + y'_i + \alpha y'_{i+1} = b\frac{y_{i+2} - y_{i-2}}{4h} + a\frac{y_{i+1} - y_{i-1}}{2h} \tag{2.1}$$

for $i = 2, 3, ..., n-1$
We can obtain the value of α, a, b by matching the Taylor's expansion of (2.1) up to $o(h^4)$. This makes the resulting scheme to be fourth-order accurate.

Doing this, we have:
$\alpha = \frac{1}{4}$, $a = \frac{3}{2}$, $b = 0$
and

$$\frac{1}{4}y'_{i-1} + y'_i + \frac{1}{4}y'_{i+1} = \frac{3(y_{i+1} - y_{i-1})}{4h} \tag{2.2}$$

(2.2) is the fourth-order Compact Finite Difference Scheme. (Li and Chen [16])

2.1 Derivation of the one sided schemes

Let us assume that a real valued function $y(x)$ and its first derivative at the boundary point satisfies the following equation:
At the point $i = n$

$$y'_n + \alpha y'_{n-1} = \frac{1}{h}\left(ay_n + by_{n-1} + cy_{n-2} + dy_{n-3}\right) \tag{2.3}$$

α, a, b, c, d can be obtained by following the same procedure as we did for (2.1), which gives:
$\alpha = 3$, $a = \frac{17}{6}$, $b = -\frac{3}{2}$, $c = -\frac{3}{2}$, $d = \frac{1}{6}$
Substituting the above into (2.3) we have:

$$3y'_{n-1} + y'_n = \frac{17}{6h}y_n - \frac{3}{2h}\left(y_{n-1} + y_{n-2}\right) + \frac{1}{6h}y_{n-3} \tag{2.4}$$

which is the fourth-order one sided boundary scheme for $i = n$. (Lele [17])

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3 Numerical Examples

The fourth-order Compact Finite Difference Scheme and the one-sided scheme obtained in section 2 is applied to solve some selected problems in the literature. Note that we use the following notations:

- CFDS:Compact Finite Difference Scheme
- FORKM:Fourth-Order Runge-Kutta Method
- AE:Absolute Error(i.e $|y_{exact} - y_{approx}|$)

3.1 Algorithm for the Implementation

Below are the steps for implementing the Compact Finite Difference Scheme together with the one sided scheme on any first order initial value problems of ordinary differential equations:

1. Input the step size h.
2. Input the number of grid points n.
3. Input the initial value for x i.e x_0.
4. Input the initial value for y i.e y_0.
5. For i from 1 to n by 1
   do
   $x_i = x_{i-1} + 0.1$
   end do
6. For i from 1 to $n-1$ by 1
   do
   $eqn_i := \frac{1}{4}f(x_{i-1}, y_{i-1}) + f(x_i, y_i) + \frac{1}{4}f(x_{i+1}, y_{i+1}) = \frac{3}{4h}(y_{i+1} - y_{i-1})$
   end do
7. Define
   $eqn_n := 3f(x_{n-1}, y_{n-1}) + f(x_n, y_n) = \frac{17}{6h}y_n - \frac{3}{2h}(y_{n-1} + y_{n-2}) + \frac{1}{6h}y_{n-3}$
8. Define
   $sys := [eqn_1, eqn_2, ..., eqn_n]$
9. Define
   variables:= $[y_1, y_2, ..., y_n]$
10. Generate the augumented matrix and assign them to A and b.
    $A, b = GenerateMatrix(sys, var)$
11. Solve for $[y_1, y_2, ..., y_n]$ using the LinearSolve command.
    LinearSolve(A,b)

Note: This algorithmn is written using the commands of the Maple software. But it can be easily translated to the commands of any other computing software.

3.2 Example 1

$y' = x - y$, for $x \in [0, 2]$, $h = 0.2$, and $y(0) = 1$
Exact solution: $y(x) = x - 1 + 2e^{-x}$
Source: Dhokrat [18].
Following the algorithmn in section 3.1 we obtained the following result and we compared our result with the result obtained by applying the fourth-order Runge-Kutta method to the same problem in Dhokrat [7]. Additionally, we plotted our result and the exact solution on the same graph.

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Table 1: Results and Absolute errors for example 1

x	CFDS	FORKM [7]	Exact	AE of CFDS	AE of FORKM
0	1	1	1	0	0
0.2	0.837459911	0.837467	0.837461506	1.595×10^{-6}	5.494×10^{-6}
0.4	0.740635022	0.740649	0.740640092	5.070×10^{-6}	8.908×10^{-6}
0.6	0.697618667	0.697634	0.697623272	4.605×10^{-6}	1.073×10^{-5}
0.8	0.698650978	0.698669	0.698657928	6.9504×10^{-6}	1.107×10^{-5}
1.0	0.735753259	0.73577	0.735758824	5.2936×10^{-6}	1.158×10^{-5}
1.2	0.802381208	0.8024	0.8023884238	7.2154×10^{-6}	1.111×10^{-5}
1.4	0.893189088	0.893205	0.8931939278	4.8397×10^{-6}	1.107×10^{-5}
1.6	1.003786285	1.0038	1.003793036	6.7510×10^{-6}	6.9640×10^{-6}
1.8	1.130593881	1.13061	1.130597776	3.8954×10^{-6}	1.222×10^{-5}
2.0	1.270664529	1.27068	1.270670566	6.0374×10^{-6}	9.4336×10^{-6}

3.3 Example 2

$y' = x^2 + xy$, for $x \in [0, 1]$, $h = 0.1$, and $y(0) = 1$
Exact solution: $y(x) = \sqrt{\frac{\pi}{2}} e^{\frac{x^2}{2}} erf(\frac{x}{\sqrt{2}}) + e^{\frac{x^2}{2}} - x$
Source: Islam [19].
Following the algorithmn in section 3.1 we obtained the following result and we plotted the result and the exact solution on the same graph.

Table 2: Results and Absolute errors for example 2

x	CFDS	Exact	AE
0	1	1	0
0.1	1.005331884	1.005346522	1.46×10^{-5}
0.2	1.022890361	1.022889463	8.99×10^{-7}
0.3	1.055178519	1.055191964	1.34×10^{-5}
0.4	1.105321051	1.105318953	2.10×10^{-6}
0.5	1.176963485	1.176974973	1.15×10^{-5}
0.6	1.274683229	1.274678992	4.24×10^{-6}
0.7	1.403980411	1.403988318	7.91×10^{-6}
0.8	1.571796300	1.571787770	8.53×10^{-6}
0.9	1.786664838	1.786665854	1.02×10^{-6}
1.0	2.059424809	2.059407405	1.74×10^{-5}

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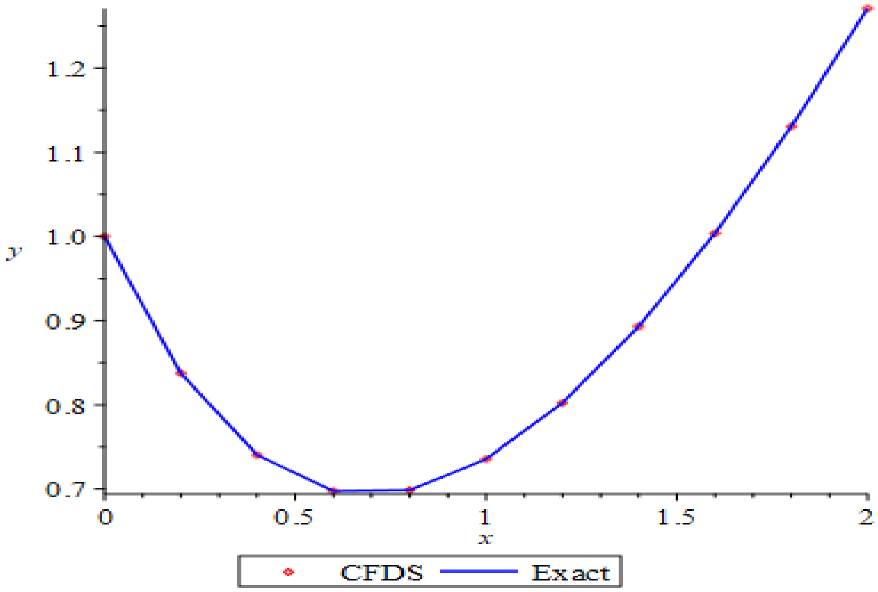

Figure 1: A graph showing the physical behavior of the solutions of example 1 on the considered domain

3.4 Example 3

$y' = -y$, for $x \in [0, 1]$, $h = 0.1$, and $y(0) = 1$
Exact solution: $y(x) = e^{-x}$
Source: Atteh and Edogbanya [1]
Following the algorithmn in section 3.1 we obtained the following result and we plotted the result and the exact solution on the same graph.

Table 3: Results and Absolute errors for example 3

x	CFDS	Exact	AE
0	1	1	0
0.1	0.904837418	0.904837433	1.48×10^{-8}
0.2	0.818730753	0.818730654	9.93×10^{-8}
0.3	0.740818221	0.740818159	6.15×10^{-8}
0.4	0.670320046	0.670319881	1.646×10^{-7}
0.5	0.606530660	0.606530551	1.083×10^{-7}
0.6	0.548811636	0.548811431	2.053×10^{-7}
0.7	0.496585304	0.496585170	1.339×10^{-7}
0.8	0.449328964	0.449328736	2.281×10^{-7}
0.9	0.406569660	0.406569516	1.441×10^{-7}
1.0	0.367879441	0.367879203	2.386×10^{-7}

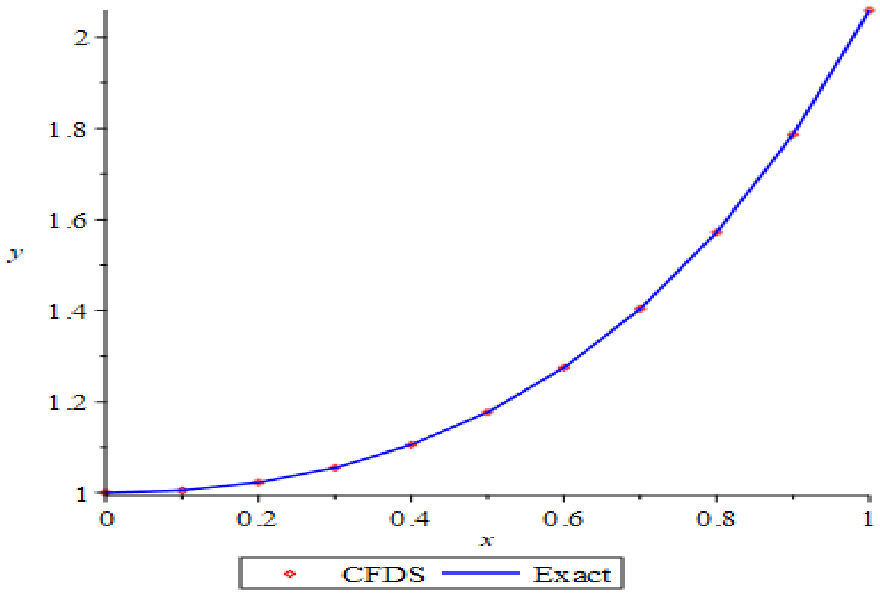

Figure 2: A graph showing the physical behavior of the solutions of example 2 on the considered domain

4 Result Discussion

Table 1 presents the results obtained using CFDS to solve Example 1, alongside those from FORKM in Dhokrat [18], along with their absolute errors. The table clearly demonstrates that CFDS, in just one iteration, produces more accurate results at all grid points compared to FORKM, which requires nine iterations. Tables 2 and 3 present the results obtained using CFDS to solve Examples 2 and 3, respectively, along with their absolute errors. The results clearly show a strong agreement with the exact solution. Figures 1, 2, and 3 display the plots of the results obtained using CFDS to solve Examples 1, 2, and 3, along with the exact solution plots. The graphs indicate that at all grid points, the numerical results closely align with the exact solution.

5 Conclusion

A fourth-order Compact Finite Difference Scheme has been presented for the numerical solution of first-order initial value problems of ordinary differential equations. The numerical results for Example 1 demonstrate that the proposed scheme achieves higher accuracy compared to the fourth-order Runge-Kutta method. Additionally, the results for Examples 2 and 3 closely match their exact solutions, confirming the scheme's computational efficiency. Notably, in all three examples, the solution at every grid point was obtained in a single iteration, highlighting the scheme's ability to save computational time.

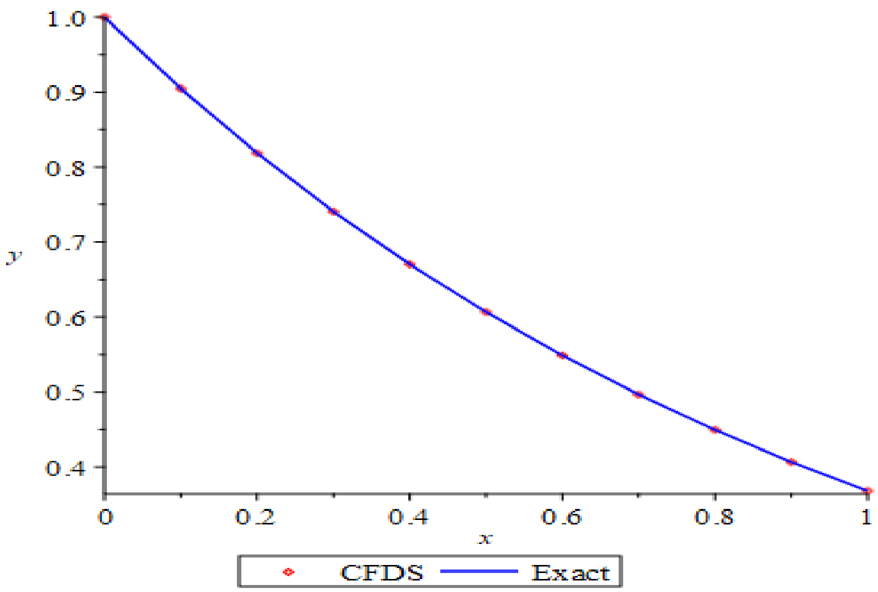

Figure 3: A graph showing the physical behavior of the solutions of example 3 on the considered domain

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