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An Inventory Model for Non-Instantaneous Deteriorating Items Under Two-Phase Demand and Two-Level Trade Credit

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Abstract

Trade credit is widely used in modern business transactions as it provides alternative to price reduction, encourage retailer's demand and minimizes holding cost. In this study, an inventory model for non-instantaneously deteriorating items under trade credit is developed. The demand is in two–phase, in the first phase, when there is no deterioration, the demand is stock dependent due to freshness of the stocked items whereas in the second phase, when deterioration sets in, the demand is assumed to be price dependent as a result of reduction in quality of the product. Profit functions of the model was obtained. The necessary and sufficient conditions for the existence and uniqueness of the optimal solutions to the profit functions was established. The Newton-Raphson iterative method was employed to find the solutions to the numerical examples using MATLAB. Sensitivity analysis was carried out to test the sensitivity of the model's parameters on the model. The major findings reveal that the holding cost, deterioration cost and the interest charged rate significantly influence the optimal cycle period and the maximum total profit.

Keywords: Non-instantaneous deterioration, Two-level trade credit, Two-phase demand.
MSC2010: 65M06, 65N35, 35F50.

1 Introduction

Inventory models for deteriorating items have received much attention from researchers in the past years. Most physical goods such as drugs, vegetables deteriorate over time [1]. Deterioration is the damage, spoilage, dryness, vaporization, etc., that result in the decrease of usefulness of the commodity. Ghare and Schrader [2] were the first to consider the optimal ordering policies for deteriorating items. They presented an EOQ model for an exponentially deteriorating item. The work was extended by Covert and Philip [3] who developed an EOQ model for a variable rate of deterioration. Shah [4] generalized the work of Ghare and Shrader [2] to allow for backordering. Hollier and Mark [5] developed a model for inventory replenishment policies for deteriorating items in a declining market. Cheng and Chen [6] considered deteriorating items in a periodic review

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environment with shortages. Recently, Macías-López *et al.* [7] studied an inventory model with a shelf space constrained and nonlinear holding cost for a perishable item.
In the literature of deteriorating items, the researchers assume that the deterioration of the items is instant on their arrival in the warehouse. But in reality, most goods would have a span of maintaining quality or original condition before they start to deteriorate. Wu *et al.* [8] defined such a phenomenon as "non-instantaneous deterioration" or delayed deteriorating [11]. Wu *et al.* [8] developed a replenishment policy for non-instantaneous deteriorating items with stock dependent demand and partial backlogging. Ouyang *et al.* [9] proposed an inventory model for non-instantaneously deteriorating items with permissible delay in payments. In the work, the demand rate is constant and shortages are not allowed. Maihami and Kamalabadi [10] developed a model for joint pricing and inventory control of non-instantaneous deteriorating item that allowed shortages with the unsatisfied demand being partially backlogged. Musa and Sani [11] considered delayed deteriorating items having two–phase demand. In the two phases, the demands of the item are both constants but different. Of recent, Liao *et al.* [12] addressed an EOQ inventory model with a delay in payment policy for non-instantaneous deteriorating items with the aim of finding an optimal ordering policy.
In the traditional inventory model, it is assumed that the supplier will receive payment for the goods as soon as they are delivered. In practice, the supplier may grace the retailer with a permissible delay in payment period. The retailer may accrue sales during this credit period and get interest on those sales. But beyond the allowed period, the supplier adds interest to the outstanding debt. Goyal [13] developed an EOQ model under the condition of a permissible delay in payments. Aggarwal and Jaggi [14] then extended Goyal's model to allow for deteriorating items under permissible delay in payments. Aliyu and Sani [15] considered two–level trade credit in developing model for deteriorating items.
Nowadays it is observed that putting massive displays of consumer goods in supermarkets raises demand and attracts in more shoppers. As a result, when choosing the optimum inventory policy, the impact of stock dependent demand cannot be disregarded. In this line, Hou and Lin [16] considered an EOQ inventory model for deteriorating items with price-and-stock dependent demand, were shortages and full backordering was considered. Agi and Soni [17] developed an inventory model for joint optimal pricing and inventory management for a perishable item under stock-, age-, and price-dependent demand, allowing surplus inventory at the end of the cycle. Idowu *et al.* [18] addressed the JIT scheduling problem on flow shop where jobs incur penalties if they are not completed within their specific due windows. Kwaghkor *et al.* [19] derived a two-state stochastic model from the interval transition probability of a Semi-Markov model used to study the transition rae between two labour market states.
This research work contributes to the field of inventory by providing an EOQ model for non-instantaneous deteriorating items, considering two–phase demand and incorporating a two–level trade credit. The model expands upon existing literature and provides insights into optimizing inventory management in situations involving non-instantaneous deterioration.

2 Notation and assumptions

The notation used in the model are:

c, p: The purchasing cost and selling price per unit item, where $p > c$.
h: The holding cost per unit item per unit time excluding the capital cost.
θ: The constant deterioration rate of the stocked item.
t_1, T: The beginning time of the item deterioration and the replenishment cycle ($T > t_1$).
$I_1(t)$: The inventory level during the period $[0, t_1]$.
$I_2(t)$: The inventory level during the period $[t_1, T]$.
I_0: The maximum inventory level/stocking capacity of the warehouse.
I_e, I_p: The retailer's interest earned and interest charged per unit item.
M: The retailer's trade credit period offered by the supplier.
N: The customer's trade credit period offered by the retailer.

A: The inventory ordering cost.

The assumptions used in building the model are:

i. A single non-instantaneous deteriorating item is considered.

ii. The replenishment rate is infinite.

iii. The lead time (the length of time between placement and receipt of an order) is zero.

iv. The demand rate in the first phase is stock dependent, i.e. $D(I(t), t) = a + bI(t)$, whereas in the second phase, the demand depends on the price, i.e. $D(p, t) = a + bp$, where $a > 0$ is the initial demand and $b > 0$ is the demand rate.

v. Shortages are not allowed.

vi Interest charged is assumed to be higher than the interest earned ($I_p > I_e$). This serves as a penalty whenever the retailer fails to settle the account as at when agreed.

vii. We restrict $N < M$ and also $N < t_1$ for convenience.

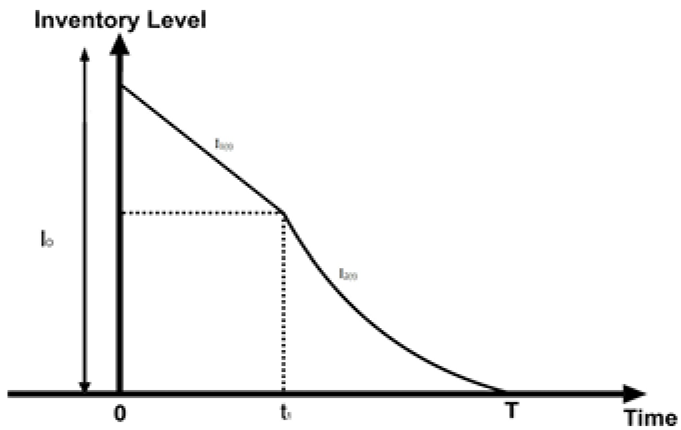

Fig. 1. Pictorial representation of the model.

3 The model formulation

At the beginning, I_0 units of item are stocked. During the time interval $[0, t_1]$, the inventory level decreases due to demand only. Subsequently the inventory level drops to zero due to both demand and deterioration during the time interval $[t_1, T]$.This phenomenon is represented as follows:

$$\frac{dI_1(t)}{dt} = -D(I_1(t), t) = -(a + bI_1(t)), \qquad 0 \le t \le t_1 \qquad (1)$$

with the initial condition $I_1(0) = I_0$
and

$$\frac{dI_2(t)}{dt} + \theta I_2(t) = -D(p, t) = -(a + bp) \qquad t_1 \le t \le T \qquad (2)$$

with the boundary condition $I_2(T) = 0$
The solutions to equations (1) and (2) are respectively given as:

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$$I_1(t) = \frac{a}{b}(e^{-bt} - 1) + I_0 e^{-bt}, \qquad 0 \le t \le t_1 \tag{3}$$

$$I_2(t) = \frac{(a+bp)}{\theta}\left(e^{\theta(T-t)} - 1\right), \qquad t_1 \le t \le T \tag{4}$$

For continuity, we let $I_1(t_1) = I_2(t_1)$. Using equations (3) and (4), I_0 can be obtained as:

$$I_0 = \frac{(a+bp)e^{bt_1}}{\theta}\left[e^{\theta(T-t_1)} - 1\right] + \frac{a}{b}(e^{bt_1} - 1) \tag{5}$$

Substituting (5) into (3) gives

$$I_1(t) = \frac{(a+bp)e^{b(t_1-t)}}{\theta}\left[e^{\theta(T-t_1)} - 1\right] + \frac{a}{b}(e^{b(t_1-t)} - 1), \qquad 0 \le t \le t_1 \tag{6}$$

To get the total profit per cycle (denoted by TP), we obtain the following:

i. Ordering Cost

The annual ordering cost *(OC)* is

$$OC = \frac{A}{T} \tag{7}$$

ii. Holding Cost

The annual holding cost *(HC)* is given by

$$HC = \frac{h}{T}\left[\int_0^{t_1} I_1(t)dt + \int_{t_1}^{T} I_2(t)dt\right] = \frac{h}{T}[(e^{bt_1} - 1)\left(\frac{(a+bp)(e^{\theta(T-t_1)} - 1)}{b\theta} + \frac{a}{b^2}\right) + t_1\left(\frac{a+bp}{\theta} - \frac{a}{b}\right) + \frac{a+bp}{\theta^2}\left(\left(e^{\theta(T-t_1)} - 1\right) - \theta T\right)]8 \tag{3.1}$$

iii. Deterioration Cost

The annual deterioration cost *(DC)* is given by

$$DC = \frac{c\theta}{T}\int_{t_1}^{T} I_2(t)dt = \frac{c(a+bp)}{T}\left[\frac{\left(e^{\theta(T-t_1)} - 1\right)}{\theta} + (t_1 - 1)\right] \tag{9}$$

iv. Sales Revenue

The annual sales revenue *(SR)* is given by

$$SR = \frac{p}{T}\left[\int_0^{T} D\,(t)\,dt\right] = \frac{p}{T}\left[\int_0^{t_1} D_1\,(t)\,dt + \int_{t_1}^{T} D_2\,(t)\,dt\right] = \frac{p}{T}[(e^{bt_1} - 1)\,(\frac{(a+bp)(e^{\theta(T-t_1)} - 1)}{\theta} + \frac{a}{b}) + (a+bp)\,(T - t_1)]10 \tag{3.2}$$

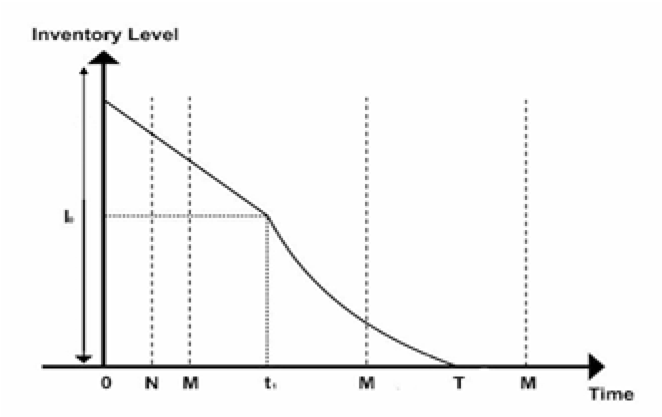

Fig. 2. Positions of N, M, t_1 and T in the model

v. Annual interest earn and interest payable

To calculate the interest earn and interest payable by the retailer with respect to M, N and T, and restricting $N < M$ and $N < t_1$, the following 3 cases may arise as shown in Figure 2:

1) $N < M < t_1$
2) $t_1 < M < T$
3) $T < M$.

Case $(1) : N < M < t_1$
After the allowed trade credit period M, the retailer must pay interest on all unsold items. Therefore, the interest payable by the retailer using equations (4) and (6), is given by:

$$IP_1 = \frac{cI_p}{T}\int_M^T I(t)dt = \frac{cI_p}{T}\left[\int_M^{t_1} I_1(t)\,dt + \int_{t_1}^T I_2(t)\,dt\right]$$

$$= \frac{cI_p}{T}[(e^{b(t_1-M)}-1)\left(\frac{(a+bp)\left(e^{\theta(T-t_1)}-1\right)}{b\theta}+\frac{a}{b^2}\right)+\frac{a}{b}(M-t_1)+\frac{(a+bp)}{\theta^2}\times$$
$$\left(e^{\theta(T-t_1)}-1+\theta(t_1-T)\right)]11 \tag{3.3}$$

The retailer will earn interest on the sales revenue during the period $[N, M]$, and is given by:

$$IE_1 = \frac{pI_e}{T}\int_N^M D_1(t)\,dt = \frac{pI_e}{T}\int_N^M (a+bI_1(t))\,dt$$

$$= \frac{pI_e}{T}\left[\left(e^{b(t_1-N)}-e^{b(t_1-M)}\right)\left(\frac{(a+bp)\left(e^{\theta(T-t_1)}-1\right)}{\theta}+\frac{a}{b}\right)\right] \tag{12}$$

Combining the above results, the retailer's annual total profit can be expressed as follows using equations $(7), (8), (9), (10), (11)$ and (12) :

$$TP_1 = \frac{1}{T}\{\left(\frac{(a+bp)\left(e^{\theta(T-t_1)}-1\right)}{\theta}+\frac{a}{b}\right)((p-\frac{h}{b})\left(e^{bt_1}-1\right)-\frac{cI_p}{b}(e^{b(t_1-M)}-1)+$$

$$pI_e\left(e^{b(t_1-N)}-e^{b(t_1-M)}\right))+(a+bp)(p\,(T-t_1)-\frac{\left(e^{\theta(T-t_1)}-1\right)}{\theta^2}(h+c\theta+cI_p)-$$

$$\frac{1}{\theta}(ht_1+cI_p\,(t_1-T)+c\theta(t_1-1)))-A-h\theta T+\frac{a}{b}(ht_1-cI_p\,(M-t_1)\} \tag{13}$$

Case $(2): t_1 < M < T$

In this case the retailer has to pay the interest for the unsold items after the time M. Therefore, the interest payable, using equation (4), is given by:

$$IP_2=\frac{cI_p}{T}\int_M^T I_2(t)dt=\frac{cI_p}{T}\left[\frac{(a+bp)}{\theta^2}\left(e^{\theta(T-M)}-1+\theta\,(M-T)\right)\right] \tag{14}$$

The retailer will earn interest on the sales revenue generated during the period $[N, M]$ and is given by:

$$IE_2=\frac{pI_e}{T}\left[\int_N^{t_1}(a+bI_1(t))\,dt+\int_{t_1}^{M}(a+bp)\,dt\right]$$

$$=\frac{pI_e}{T}\left[\left(\frac{(a+bp)\left(e^{\theta(T-t_1)}-1\right)}{\theta}+\frac{a}{b}\right)\left(e^{b(t_1-N)}-1\right)+(a+bp)\,(M-t_1)\right] \tag{15}$$

Therefore, the retailer's total profit in this case using equations $(7),(8),(9),(10),(14)$ and (15) is given by:

$$TP_2=\frac{1}{T}\{\left(\frac{(a+bp)\left(e^{\theta(T-t_1)}-1\right)}{\theta}+\frac{a}{b}\right)((p-\frac{h}{b})\left(e^{bt_1}-1\right)+pI_e\left(e^{b(t_1-N)}-1\right))+(a+bp)\times$$

$$(p\,((T-t_1)+I_e\,(M-t_1))-\frac{\left(e^{\theta(T-t_1)}-1\right)}{\theta^2}(h+c\theta)-\frac{cI_p}{\theta^2}\left(e^{\theta(T-M)}-1+\theta\,(M-T)\right)-$$

$$\frac{1}{\theta}(ht_1+cI_p\,(t_1-T)+c\theta(t_1-1)))-A-h(\theta T+\frac{at_1}{b})\} \tag{16}$$

Case $(3): T < M$

In this case, the items have been sold before the time M. Therefore, the retailer will pay no interest, that is:

$$IP_3=0 \tag{17}$$

The annual interest earn by the retailer is given by:

$$IE_3=\frac{pI_e}{T}\left[\int_N^{t_1}(a+bI_1(t))\,dt+\int_{t_1}^{T}(a+bp)\,dt+(a+bp)\,(M-T)\right]$$

$$=\frac{pI_e}{T}\left[\left(\frac{(a+bp)\left(e^{\theta(T-t_1)}-1\right)}{\theta}+\frac{a}{b}\right)\left(e^{b(t_1-N)}-1\right)+(a+bp)\,(M-t_1)\right] \tag{18}$$

Therefore, the retailer's total profit in this case using equations $(7),(8),(9),(10),(17)$ and (18) is given by:

$$TP_3=\frac{1}{T}\{\left(\frac{(a+bp)\left(e^{\theta(T-t_1)}-1\right)}{\theta}+\frac{a}{b}\right)((p-\frac{h}{b})\left(e^{bt_1}-1\right)+pI_e\left(e^{b(t_1-N)}-1\right))+(a+bp)\times$$

$$(p\,((T-t_1)+I_e\,(M-t_1))-\frac{\left(e^{\theta(T-t_1)}-1\right)}{\theta^2}(h+c\theta)-\frac{1}{\theta}(ht_1+c\theta(t_1-1)))-A-h(\theta T+\frac{at_1}{b})\} \tag{19}$$

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4 Optimization and analysis

Since we are dealing with two decision variables t_1 and T, we use multivariable optimization approach.
The necessary conditions for TP_1 to be maximized are $\frac{\partial TP_1}{\partial t_1} = 0$ and $\frac{\partial TP_1}{\partial T} = 0$. That is, using equation (13) ,

$$\frac{\partial TP_1}{\partial t_1} = \frac{1}{T}\{\frac{a+bp}{\theta}\{p\left((b-\theta)e^{\theta(T-t_1)+bt_1} - \theta\left(e^{\theta(T-t_1)}+1\right)+be^{bt_1}\right) - \frac{h}{b}((be^{\theta(T-t_1)}+e^{bt_1}\times$$

$$\left(e^{\theta(T-t_1)}-1\right)+1) - \theta\left(e^{\theta(T-t_1)}\left(e^{bt_1}-1\right)\right) - c\theta(1-e^{\theta(T-t_1)}) - \frac{cI_p}{b}(be^{b(t_1-M)}\left(e^{\theta(T-t_1)}-1\right)+$$

$$\left(1-e^{\theta(T-t_1)}\right)) - \theta\left(e^{\theta(T-t_1)}\left(e^{b(t_1-M)}-1\right)\right) + \frac{a}{b}(pbe^{bt_1} - h(e^{bt_1}-1) - cI_p(e^{b(t_1-M)}-1))+$$

$$pI_e[\left(a+\frac{b\,(a+bp)\left(e^{\theta(T-t_1)}-1\right)}{\theta}\right)\left(e^{b(t_1-N)}-e^{b(t_1-M)}\right)+e^{\theta(T-t_1)}\,(a+bp)\,(e^{b(t_1-M)}-$$

$$e^{b(t_1-N)})]\} = 0 \tag{20}$$

and

$$\frac{\partial TP_1}{\partial T} = \frac{1}{T}\{(a+bp)\,(p(\left(1-e^{\theta(T-t_1)}\left(1-e^{bt_1}\right)\right) - h(e^{\theta(T-t_1)}\left(\frac{1}{\theta}+\frac{e^{bt_1}-1}{b}\right)-\frac{1}{\theta})-$$

$$c(e^{\theta(T-t_1)}-1) - cI_p(\frac{\left(e^{\theta(T-t_1)}\right)\left(e^{b(t_1-M)}-1\right)}{b}+\frac{\left(e^{\theta(T-t_1)}\right)}{\theta}) + pI_e(e^{\theta(T-t_1)}\times$$

$$\left(e^{b(t_1-N)}-e^{b(t_1-M)}\right))) + \theta cI_p - TP_1\} = 0 \tag{21}$$

The solutions to (20) and (21) give the values of t_1 and T.
To find the critical point $\left(t_1^{1*}, T_1^*\right)$, equations (20) and (21) are solved simultaneously. To obtain the maximum total profit, the following conditions must be satisfied for the critical point: $\frac{\partial^2 TP_1}{\partial t_1^2}.\frac{\partial^2 TP_1}{\partial T^2} - \frac{\partial^2 TP_1}{\partial t_1 \partial T}.\frac{\partial^2 TP_1}{\partial T \partial t_1} > 0$ and $\frac{\partial^2 TP_1}{\partial t_1^2} < 0$, $\frac{\partial^2 TP_1}{\partial T^2} < 0$. Thus, we give a lemma to prove some important theoretical results:

Lemma 1: If $e^{b(t_1-N)} < e^{b(t_1-M)}$ and $e^{bt_1} < 1$, then

i. $\frac{\partial^2 TP_1\left(t_1^{1*},T_1^*\right)}{\partial t_1^2} < 0$ and $\frac{\partial^2 TP_1\left(t_1^{1*},T_1^*\right)}{\partial T^2} < 0$.

ii. $\frac{\partial^2 TP_1\left(t_1^{1*},T_1^*\right)}{\partial t_1^2} > \frac{\partial^2 TP_1\left(t_1^{1*},T_1^*\right)}{\partial t_1 \partial T}$ and $\frac{\partial^2 TP_1\left(t_1^{1*},T_1^*\right)}{\partial T^2} > \frac{\partial^2 TP_1\left(t_1^{1*},T_1^*\right)}{\partial T \partial t_1}$

Proof:

i. We have

$$\frac{\partial^2 TP_1\left(t_1^{1*},T_1^*\right)}{\partial t_1^2} = \frac{1}{T}\{p[I_e\left(e^{b(t_1-N)}-e^{b(t_1-M)}\right)(b(a+\frac{b\,(a+bp)\left(e^{\theta(T-t_1)}-1\right)}{\theta})+$$

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$$e^{\theta(T-t_1)}\left(a+bp\right)\left(\theta-2b\right))+(abe^{bt_1}+\frac{a+bp}{\theta}(e^{\theta(T-t_1)}((b-\theta)^2\, e^{bt_1}+\theta^2)+b^2e^{bt_1}))]-$$

$$c\left(a+bp\right)\left[\theta e^{\theta(T-t_1)}\right]-h[\frac{a+bp}{b\theta}(b\left((b-\theta)\, e^{\theta(T-t_1)+bt_1}-be^{bt_1}-\theta e^{\theta(T-t_1)}\right)-\theta((b-$$

$$\theta)e^{\theta(T-t_1)+bt_1}+\theta e^{\theta(T-t_1)}))+ae^{bt_1}]-cI_p[\frac{a+bp}{b\theta}(b^2e^{b(t_1-M)}\left(e^{\theta(T-t_1)}-1\right)+(e^{\theta(T-t_1)}-$$

$$1)+\theta^2e^{\theta(T-t_1)}\left(e^{b(t_1-M)}-1\right)+b\theta e^{\theta(T-t_1)}\left(1-2e^{b(t_1-M)}\right))+ae^{b(t_1-M)}]\}$$

From hypothesis, if $e^{b(t_1-N)}<e^{b(t_1-M)}$ then $e^{b(t_1-N)}-e^{b(t_1-M)}<0$, and so $I_e(e^{b(t_1-N)}-e^{b(t_1-M)})(b(a+\frac{b(a+bp)\left(e^{\theta(T-t_1)}-1\right)}{\theta})+e^{\theta(T-t_1)}\left(a+bp\right)\left(\theta-2b\right))<0$. Also if $e^{bt_1}<1$, then $(abe^{bt_1}+\frac{a+bp}{\theta}(e^{\theta(T-t_1)}((b-\theta)^2\, e^{bt_1}+\theta^2)+b^2e^{bt_1}))<0$. Therefore all the terms in the equation are negative. Sum of negative terms is negative. Hence $\frac{\partial^2TP_1\left(t_1^{1*},T_1^*\right)}{\partial t_1^2}<0$.
Also,

$$\frac{\partial^2TP_1\left(t_1^{1*},T_1^*\right)}{\partial T^2}=\frac{1}{T}\{p[\theta\left(a+bp\right)\left(e^{\theta(T-t_1)}\right)((I_e(e^{b(t_1-N)}-e^{b(t_1-M)}))+\left(e^{bt_1}-1\right))]-$$

$$h[e^{\theta(T-t_1)}\left(a+bp\right)\left(1+\frac{\theta}{b}\left(e^{bt_1}-1\right)\right)]-c\left(a+bp\right)\left[\theta e^{\theta(T-t_1)}\right]-cI_p[e^{\theta(T-t_1)}\left(a+bp\right)\times$$

$$(1+\frac{\theta}{b}\left(e^{(bt_1-M)}-1\right))]\}$$

Similarly from the hypothesis, it is seen that $\frac{\partial^2TP_1\left(t_1^{1*},T_1^*\right)}{\partial T^2}<0$.

ii. Let $F(t_1)=\frac{\partial^2TP_1\left(t_1^{1*},T_1^*\right)}{\partial t_1^2}-\frac{\partial^2TP_1\left(t_1^{1*},T_1^*\right)}{\partial t_1\partial T}$. We show that $F(t_1)>0$. Thus we have

$$F(t_1)=\frac{1}{T}\{(a+bp)(e^{\theta(T-t_1)})[(\frac{b}{\theta}-1)(pbe^{bt1}+he^{bt1}-cI_pe^{b(t_1-M)})+\frac{1}{b}(h(1-\theta)+$$

$$cI_p(\frac{1}{\theta}-\theta))+pI_e((e^{b(t_1-M)}-e^{b(t_1-N)})(\frac{b^2}{\theta}-\theta+2b))]+\frac{a+bp}{\theta}[-pb^2e^{bt1}-he^{bt1}+$$

$$cI_p(be^{b(t_1-M)}+1)+pI_e(b^2(e^{b(t_1-M)}-e^{b(t_1-N)}))]-ae^{bt1}(pb-h)+cI_p(ae^{b(t_1-M)})+$$

$$pI_e(ab(e^{b(t_1-M)}-e^{b(t_1-N)}))\}$$

Since $T>t_1$, then $e^{\theta(T-t_1)}>0$. From the hypothesis, if $e^{bt1}<1$ and assuming $\frac{b}{\theta}<1$ then $(\frac{b}{\theta}-1)(pbe^{bt1}+he^{bt1}-cI_pe^{b(t_1-M)})>0$, $\frac{1}{b}(h(1-\theta)+cI_p(\frac{1}{\theta}-\theta))>0$, $\frac{a+bp}{\theta}(-pb^2e^{bt1}-he^{bt1})>0$ and $-ae^{bt1}(pb-h)>0$. Also if $e^{b(t_1-M)}-e^{b(t_1-N)}>0$, then $pI_e((e^{b(t_1-M)}-e^{b(t_1-N)})(\frac{b^2}{\theta}-\theta+2b))>0$ and $cI_p(ae^{b(t_1-M)})>0$. Hence it is seen that $F(t_1)>0$ and thus $\frac{\partial^2TP_1\left(t_1^{1*},T_1^*\right)}{\partial t_1^2}>\frac{\partial^2TP_1\left(t_1^{1*},T_1^*\right)}{\partial t_1\partial T}$.

Let $F(T)=\frac{\partial^2TP_1\left(t_1^{1*},T_1^*\right)}{\partial T^2}-\frac{\partial^2TP_1\left(t_1^{1*},T_1^*\right)}{\partial T\partial t_1}$. We show that $F(T)>0$. We have

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$$F(T)=\frac{1}{T}\{(a+bp)(e^{\theta(T-t_1)})[p(e^{bt1}(\theta-b-1)+(1+\theta))+h(2-e^{bt1})+cI_p(e^{b(t_1-M)})+$$

$$pI_e(b(e^{b(t_1-M)}-e^{b(t_1-N)}))]\}$$

Similarly we have $p(e^{bt1}(\theta-b-1)+(1+\theta))+h(2-e^{bt1})+cI_p(e^{b(t_1-M)})+pI_e(b(e^{b(t_1-M)}-e^{b(t_1-N)}))]>0$. Therefore $\frac{\partial^2 TP_1(t_1^{1*},T_1^*)}{\partial T^2}>\frac{\partial^2 TP_1(t_1^{1*},T_1^*)}{\partial T\partial t_1}$

Theorem 1: If the Hessian Matrix for equation (13) is negative definite, then TP_1 is a concave function.

Proof:

Lemma 1 confirms that the required Hessian Matrix is negative definite, since $\frac{\partial^2 TP_1(t_1^{1*},T_1^*)}{\partial t_1^2}>\frac{\partial^2 TP_1(t_1^{1*},T_1^*)}{\partial t_1\partial T}$ and also $\frac{\partial^2 TP_1(t_1^{1*},T_1^*)}{\partial T^2}>\frac{\partial^2 TP_1(t_1^{1*},T_1^*)}{\partial T\partial t_1}$. Thus the determinant $\frac{\partial^2 TP_1(t_1^{1*},T_1^*)}{\partial t_1^2}.\frac{\partial^2 TP_1(t_1^{1*},T_1^*)}{\partial T^2}-\frac{\partial^2 TP_1(t_1^{1*},T_1^*)}{\partial t_1\partial T}.\frac{\partial^2 TP_1(t_1^{1*},T_1^*)}{\partial T\partial t_1}>0$. Hence TP_1 is concave.

Similar approach is applied for TP_2 and TP_3 respectively.

The necessary conditions for TP_2 to be maximized are $\frac{\partial TP_2}{\partial t_1}=0$ and $\frac{\partial TP_2}{\partial T}=0$. That is, using equation (16) :

$$\frac{\partial TP_2}{\partial t_1}=\frac{1}{T}\{\frac{a+bp}{\theta}\{p\left((b-\theta)e^{\theta(T-t_1)+bt_1}-\theta\left(e^{\theta(T-t_1)}+1\right)+be^{bt_1}\right)-\frac{h}{b}((be^{\theta(T-t_1)}+e^{bt_1}\times$$

$$\left(e^{\theta(T-t_1)}-1\right)+1)-\theta\left(e^{\theta(T-t_1)}\left(e^{bt_1}-1\right)\right)-c\theta(1-e^{\theta(T-t_1)})+\theta pI_e(e^{\theta(T-t_1)}(\frac{b}{\theta}e^{b(t_1-N)}-$$

$$e^{b(t_1-N)}+1)-\theta))+\frac{a}{b}(pba(e^{bt_1}+I_e e^{b(t_1-N)})-h(e^{bt_1}-1))\}=0 \tag{22}$$

and

$$\frac{\partial TP_2}{\partial T}=\frac{1}{T}\{(a+bp)\left(p(\left(1-e^{\theta(T-t_1)}\left(1-e^{bt_1}\right)\right)-h(e^{\theta(T-t_1)}\left(\frac{1}{\theta}+\frac{e^{bt_1}-1}{b}\right)-\frac{1}{\theta})-\right.$$

$$c(e^{\theta(T-t_1)}-1)-cI_p(\frac{(e^{\theta(T-M)}-1)}{\theta})+pI_e(e^{\theta(T-t_1)}\left(e^{b(t_1-N)}-1\right)))-TP_2\}=0 \tag{23}$$

The solutions to (22) and (23) give the values of t_1 and T.

For the sufficient condition, we have found that the determinant of the Hessian matrix $\begin{bmatrix}\frac{\partial^2 TP_2}{\partial t_1^2} & \frac{\partial^2 TP_2}{\partial T\partial t_1}\\ \frac{\partial^2 TP_2}{\partial t_1\partial T} & \frac{\partial^2 TP_2}{\partial T^2}\end{bmatrix}$ evaluated at $\left(t_1^{2*},T_2^*\right)$ is negative definite.

The necessary condition for TP_3 to be maximized are $\frac{\partial TP_3}{\partial t_1}=0$ and $\frac{\partial TP_3}{\partial T}=0$. That is, using equation (19) :

$$\frac{\partial TP_3}{\partial t_1}=\frac{1}{T}\{\frac{a+bp}{\theta}\{p\left((b-\theta)e^{\theta(T-t_1)+bt_1}-\theta\left(e^{\theta(T-t_1)}+1\right)+be^{bt_1}\right)-\frac{h}{b}((be^{\theta(T-t_1)}+e^{bt_1}\times$$

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$$\left(e^{\theta(T-t_1)}-1\right)+1)-\theta\left(e^{\theta(T-t_1)}\left(e^{bt_1}-1\right)\right)-c\theta(1-e^{\theta(T-t_1)})+\theta pI_e(e^{\theta(T-t_1)}(\frac{b}{\theta}e^{b(t_1-N)}-$$

$$e^{b(t_1-N)}+1)-\theta))+\frac{a}{b}(pba(e^{bt_1}+I_ee^{b(t_1-N)})-h(e^{bt_1}-1))\}=0 \tag{24}$$

and

$$\frac{\partial TP_3}{\partial T}=\frac{1}{T}\{(a+bp)\left(p(\left(1-e^{\theta(T-t_1)}\left(1-e^{bt_1}\right)\right)-h(e^{\theta(T-t_1)}\left(\frac{1}{\theta}+\frac{e^{bt_1}-1}{b}\right)-\frac{1}{\theta})-\right.$$

$$c(e^{\theta(T-t_1)}-1)+pI_e(e^{\theta(T-t_1)}\left(e^{b(t_1-N)}-1\right)))-TP_3\}=0 \tag{25}$$

The solutions to (24) and (25) give the values of t_1 and T.

For the sufficient conditions, we have found that the determinant of the Hessian matrix $\begin{bmatrix} \frac{\partial^2 TP_3}{\partial t_1^2} & \frac{\partial^2 TP_3}{\partial T\partial t_1} \\ \frac{\partial^2 TP_3}{\partial t_1\partial T} & \frac{\partial^2 TP_3}{\partial T^2} \end{bmatrix}$ evaluated at $\left(t_1^{3*}, T_3^*\right)$ is negative definite.

5 Result and analysis

Based on the analyses provided above, the following algorithm can be utilized to find the optimal solution of this model.

5.1 Algorithm

We use this algorithm for case 1, $N < M < t_1$

Step 1: Start with $k=0$ and the initial value of $X_0=(t_{1_0},T_0)$.

Step 2: By using (20) and (21), find the initial $X_0=(t_{1_0},T_0)$.

Step 3: Use the result in step 2 to determine the optimal $X_{k+1}=(t_1^*,T^*)$ by using the Newton-Raphson formula

$X_{k+1}=X_k-\left(H_f|_{(X_k)}\right)^{-1}\nabla f|_{(X_k)}$ where $H_f|_{(X_k)}=\begin{pmatrix} \frac{\partial^2 TP_1}{\partial t_1} & \frac{\partial^2 TP_1}{\partial t_1\partial T} \\ \frac{\partial^2 TP_1}{\partial T\partial t_1} & \frac{\partial^2 TP_1}{\partial T^2} \end{pmatrix}|_{(X_k)}$ and

$\nabla f|_{(X_k)}=\begin{pmatrix} \frac{\partial TP_1}{\partial t_1} \\ \frac{\partial TP_1}{\partial T} \end{pmatrix}$|_(X_k)$for$k=0,1,2,...

Step 4: If the difference between X_{k+1} and X_k is sufficiently small (i.e.,$|X_{k+1}-X_k|\leq 0.0001$), then X_{k+1} is the optimal (t_1^*,T^*), so stop. Otherwise set $X_{k+1}=X_k$ and go back to step 3.

By using above algorithm, we obtain the optimal TP_1^* by using equation (13).The algorithm is repeated for case 2 and case 3. Due to the non–linear nature of the equations, they are solved using MATLAB software.

5.2 Numerical Examples

To illustrate the solution procedure and the results, the proposed algorithm is applied to solve the following numerical examples. The graph for the total pofit is shown in Fig. 3 below.

Example 1: for case 1 ($N < M < t_1$)

Consider an inventory system with the following input parameters: $A=600, p=20, c=12, a=150, b=0.30, h=0.4, M=0.25, N=0.15, \theta=0.30, I_p=0.15, I_e=0.12$. Here we obtain as follows the values of the optimal length of time with positive inventory $t_1^{1*}=0.4802$ year (175 days), the optimal cycle length $T_1^* = 0.9088$ year (332 days), and the optimal total profit $TP_1^* = 2835.2110$ per year. Observe that the condition $N < M < t_1$ is satisfies.

Example 2: for case 2 ($t_1 < M < T$)

The data is the same as in Example 1 except that $N = 0.1$. Here we obtain as follows the values of the optimal length of time with positive inventory $t_1^{2*} = 0.1218$ year (44 days), the optimal cycle length $T_2^* = 0.7613$ year (278 days), and the optimal total profit $TP_2^* = 1249.0173$ per year. Observe that $t_1 < M < T$ which satisfies the condition.

Example 3: for case 3 ($T < M$)

The data is the same as in Example 1 except that $M = 1.0, N = 0.35$. Here we obtain as follows the values of the optimal length of time with positive inventory $t_1^{3*} = 0.4050$ year (148 days), the optimal cycle length $T_3^* = 0.9107$ year (332 days), and the optimal total profit $TP_3^* = 1061.1442$ per year. Observe that $T < M$ which satisfies the condition.

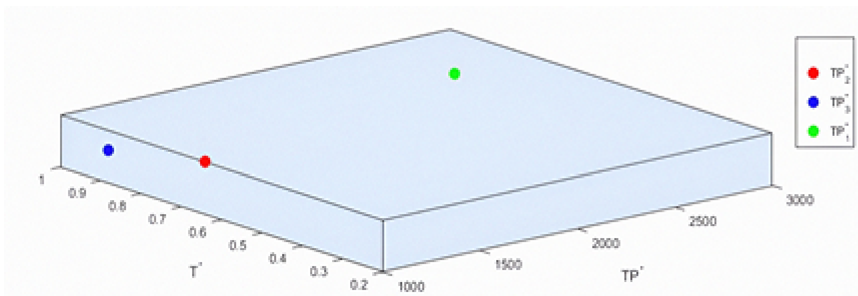

Fig. 3. Graphical representation of Total Profit.

5.3 Sensitivity Analysis

In order to illustrate the effect of the parameters on the optimal policies of the example, a sensitivity analysis is performed by changing the values of only one parameter at a time and keeping the rest of the parameters at their initial values. The results are shown in Tables 1, 2 and 3.

Table 1. Sensitivity analysis result of cases (percentage change in system parameters against values for case 1)

Parameter	*% change in parameter*	*New* t_1^*	*% in* t_1^*	*New* T^*	*% in* T^*	*New* TP^*	*% in* TP^*
A	+10	0.4883	1.6868	0.9311	2.4538	2867.3385	1.1332
	+5	0.4842	0.8330	0.9200	1.2324	2851.3457	0.5691
	+0	0.4802	0	0.9088	0	2835.2110	0
	−5	0.4761	−0.8538	0.8975	−1.2434	2818.7985	−0.5789
	−10	0.4720	−1.7076	0.8862	−2.4868	2801.9838	−1.1719
c	+10	0.4796	−0.1249	0.8694	−4.3354	2762.7695	−2.5551
	+5	0.4801	−0.0208	0.8914	−1.9146	2796.9833	−1.3483
	+0	0.4802	0	0.9088	0	2835.2110	0
	−5	0.4805	0.0625	0.9318	2.5308	2869.6389	1.2143
	−10	0.4808	0.1249	0.9574	5.3477	2921.4828	3.0429
h	+10	0.4797	−0.1041	0.8708	−4.1813	2792.1152	−1.5200
	+5	0.4801	0.0208	0.8994	−1.0343	2821.0242	−0.5004
	+0	0.4802	0	0.9088	0	2835.2110	0
	−5	0.4802	0	0.9181	1.0233	2849.7866	0.5141
	−10	0.4814	0.2499	0.9365	3.0480	2898.9016	2.2464
θ	+10	0.4244	−11.6202	0.8742	−3.8072	2520.4296	−11.1026
	+5	0.4453	−7.2678	0.8963	−1.3754	2624.8119	−7.4209
	+0	0.4802	0	0.9088	0	2835.2110	0
	−5	0.5006	4.2482	1.0324	13.6004	3117.5903	9.9597
	−10	0.5009	4.3107	1.0914	20.0924	3342.9147	17.9071
p	+10	0.5154	7.3303	1.0331	13.6774	3024.8303	6.6880
	+5	0.4659	1.1870	0.9298	2.3107	2980.8418	5.1365
	+0	0.4802	0	0.9088	0	2835.2110	0
	−5	0.4746	−1.1662	0.8876	−2.3327	2692.3423	−5.0391
	−10	0.4582	−4.5814	0.8227	−9.4740	2631.3085	−7.1918

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Parameter	*% change in parameter*	*New* t_1^*	*% in* t_1^*	*New* T^*	*% in* T^*	*New* TP^*	*% in* TP^*
a	+10	0.4745	−1.1870	0.8806	−3.1030	3137.7411	10.6705
	+5	0.4773	−0.6039	0.8945	−1.5735	2987.7583	5.3805
	+0	0.4802	0	0.9088	0	2835.2110	0
	−5	0.4824	0.4581	0.9200	1.2324	2685.1547	−5.2926
	−10	0.4847	0.9371	0.9315	2.4978	2531.8274	−10.7006
b	+10	0.3328	−30.6955	0.8087	−11.0145	3289.3211	16.0168
	+5	0.4172	−13.1195	0.8769	−3.5101	3236.1317	14.1408
	+0	0.4802	0	0.9088	0	2835.2110	0
	−5	0.4944	2.5614	0.9784	7.6585	2491.2686	−12.1311
	−10	0.4925	2.9571	1.0212	12.3680	2206.5712	−22.1726
I_p	+10	0.4830	0.5831	0.9613	5.7768	2752.9516	−2.9014
	+5	0.4819	0.3540	9345	2.8279	2794.3002	−1.4430
	+0	0.4802	0	0.9088	0	2835.2110	0
	−5	0.4775	−0.5623	0.8858	−2.5308	2872.7706	1.3248
	−10	0.4739	−1.3120	0.8606	−5.3037	2915.9077	2.8462
I_e	+10	0.4810	0.1666	0.9110	0.2421	2836.8657	0.0584
	+5	0.4806	0.0833	0.9099	0.1210	2836.0384	0.0292
	+0	0.4802	0	0.9088	0	2835.2110	0
	−5	0.4798	−0.0833	0.9077	−0.1210	2834.3833	−0.0292
	−10	0.4794	−0.1666	0.9066	−0.2421	2833.5555	−0.0584
M	+10	0.4942	2.9155	0.9562	5.2157	2859.1117	0.8430
	+5	0.4841	0.8122	0.9220	1.4525	2846.8510	0.4106
	+0	0.4802	0	0.9088	0	2835.2110	0
	−5	0.4762	−0.8330	0.8955	−1.4635	2824.0998	−0.3919
	−10	0.4722	−1.6660	0.8821	−2.9379	2783.6177	−1.8197
N	+10	0.4814	0.2499	0.9121	0.3631	2829.3608	−0.2063
	+5	0.4808	0.1249	0.9104	0.1761	2834.0033	−0.0426
	+0	0.4802	0	0.9088	0	2835.2110	0
	−5	0.4798	−0.0833	0.9077	−0.1210	2836.5971	0.0489
	−10	0.4796	−0.1249	0.9071	−0.1871	2837.8207	0.0920

Table 2. Sensitivity analysis result of cases (percentage change in system parameters against values for case 2)

Parameter	*% change in parameter*	*New* t_1^*	*% in* t_1^*	*New* T^*	*% in* T^*	*New* TP^*	*% in* TP^*
A	+10	0.1255	3.0378	0.7812	2.6139	1369.1970	9.6219
	+5	0.1236	1.4778	0.7712	1.3004	1308.3035	4.7466
	+0	0.1218	0	0.7613	0	1249.0173	0
	−5	0.1199	−1.5599	0.7513	−1.3135	1188.0843	−4.8785
	−10	0.1180	−3.1199	0.7413	−2.6271	1127.1311	−9.7586
c	+10	0.0961	−21.1002	0.7059	−7.2770	809.0986	−35.2212
	+5	0.1085	−10.9195	0.7324	−3.7961	1023.3607	−18.0667
	+0	0.1218	0	0.7613	0	1249.0173	0
	−5	0.1384	13.6289	0.7985	4.8864	1325.2235	6.1013
	−10	0.1512	24.1379	0.8279	8.7482	1534.3285	22.8429
h	+10	0.1025	−15.8456	0.7544	−0.9063	936.0068	−25.0605
	+5	0.1122	−7.8818	0.7579	−0.2102	1093.6899	−12.4360
	+0	0.1218	0	0.7613	0	1249.0173	0
	−5	0.1312	7.7176	0.7644	0.4072	1351.0756	8.1711
	−10	0.1406	15.4351	0.7675	0.8144	1400.3892	12.1193
θ	+10	0.0930	−23.6453	0.6971	−8.4329	1089.4099	−12.7786
	+5	0.0933	−23.3990	0.7339	−3.5991	1110.5268	−11.0880
	+0	0.1218	0	0.7613	0	1249.0173	0
	−5	0.1526	25.2874	0.7842	3.0080	1307.8870	4.7133
	−10	0.1749	43.5961	0.7864	3.2970	1483.5457	18.7770

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Parameter	*% change in parameter*	*New* t_1^*	*%* *in* t_1^*	*New* T^*	*%* *in* T^*	*New* TP^*	*%* *in* TP^*
p	+10	0.1323	8.6207	0.8300	9.0240	1632.0358	30.6656
	+5	0.1230	0.9852	0.7705	1.2085	1367.6253	9.4961
	+0	0.1218	0	0.7613	0	1249.0173	0
	−5	0.1163	−4.5156	0.7143	−6.1737	1065.5064	−14.6924
	−10	0.1124	−7.7176	0.6662	−12.4918	913.9849	−26.8237
a	+10	0.1202	−1.3136	0.7394	−2.8767	1405.3350	12.5153
	+5	0.1210	−0.6568	0.7510	−1.3529	1327.8283	6.3098
	+0	0.1218	0	0.7613	0	1249.0173	0
	−5	0.1226	0.6568	0.7705	1.2085	1168.9024	−6.4142
	−10	0.1236	1.4778	0.7789	2.3118	1090.4125	−12.6984
b	+10	0.0736	−39.5731	0.7121	−6.4626	1415.5333	13.3318
	+5	0.1150	−5.5829	0.7341	−3.5728	1369.7022	9.6624
	+0	0.1218	0	0.7613	0	1249.0173	0
	−5	0.1286	5.5829	0.7760	1.9309	1120.5694	−10.2839
	−10	0.1536	26.1084	0.7797	2.4169	1015.6940	−18.6805
I_p	+10	0.1230	0.9852	0.7678	0.8538	1229.4481	−1.5668
	+5	0.1222	0.3284	0.7634	0.2758	1239.2337	−0.7833
	+0	0.1218	0	0.7613	0	1249.0173	0
	−5	0.1212	−0.4926	0.7581	−0.4203	1255.5443	0.5226
	−10	0.1206	−0.9852	0.7549	−0.8407	1268.5783	1.5661
I_e	+10	0.1428	17.2414	0.7681	0.8932	1356.4450	8.6010
	+5	0.1374	12.8079	0.7619	0.0788	1305.0269	4.4843
	+0	0.1218	0	0.7613	0	1249.0173	0
	−5	0.1191	−2.2167	0.7571	−0.5517	1202.2498	−3.7443
	−10	0.0928	−23.8095	0.7525	−1.1559	1089.6043	−12.7631
M	+10	0.1231	1.0673	0.7684	0.9326	1279.0956	2.4082
	+5	0.1224	0.4926	0.7648	0.4597	1263.2656	1.1408
	+0	0.1218	0	0.7613	0	1249.0173	0
	−5	0.1211	−0.5747	0.7577	−0.4729	1233.0958	−1.2747
	−10	0.1204	−1.1494	0.7541	−0.9458	1217.1282	−2.5531
N	+10	0.1218	0	0.7633	0.2627	1245.3097	−0.2968
	+5	0.1218	0	0.7623	0.1314	1247.1621	−0.1485
	+0	0.1218	0	0.7613	0	1249.0173	0
	−5	0.1217	−0.0821	0.7603	−0.1314	1249.2476	0.0184
	−10	0.1217	−0.0821	0.7593	−0.2627	1251.1084	0.1674

Table 3. Sensitivity analysis result of cases (percentage change in system parameters against values for case 3)

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Parameter	*% change*	*New* t_1^*	*% in* t_1^*	*New* T^*	*% in* T^*	*New* TP^*	*% in* TP^*
	in parameter						
A	+10	0.4142	2.2716	0.9494	4.2495	1179.0224	11.1086
	+5	0.4097	1.1605	0.9302	2.1412	1120.7355	5.6158
	+0	0.4050	0	0.9107	0	1061.1442	0
	−5	0.4001	−1.2099	0.8906	−2.2071	1000.2422	−5.7393
	−10	0.3948	−2.5185	0.8701	−4.4581	936.784	−11.7219
c	+10	0.3761	−7.1358	0.8296	−8.9052	814.44462	−23.2483
	+5	0.3924	−3.1111	0.8750	−3.9201	951.4140	−10.3407
	+0	0.4050	0	0.9107	0	1061.1442	0
	−5	0.4216	4.0987	0.9575	5.1389	1192.6198	12.3900
	−10	0.4398	8.5926	1.0095	10.8488	1329.7648	25.3142
h	+10	0.3747	−7.4815	0.9026	−0.8894	877.9883	−17.2602
	+5	0.3900	−3.7037	0.9070	−0.4063	970.9368	−8.5010
	+0	0.4050	0	0.9107	0	1061.1442	0
	−5	0.4196	3.6049	0.9138	0.3404	1148.0286	8.1878
	−10	0.4338	7.1111	0.9165	0.6369	1231.6455	16.0677

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Parameter	*% change in parameter*	*New* t_1^*	*% in* t_1^*	*New* T^*	*% in* T^*	*New* TP^*	*% in* TP^*
θ	+10	0.3056	−24.5432	0.9095	−0.1318	748.5092	−29.4621
	+5	0.3149	−22.2469	0.9096	−0.1208	999.3627	−5.8222
	+0	0.4050	0	0.9107	0	1061.1442	0
	−5	0.4268	5.3827	0.9170	0.6918	1195.2237	12.6354
	−10	0.4911	21.2593	0.9589	5.2926	1579.9845	48.8944
p	+10	0.4227	4.3704	1.0515	15.4606	1407.1874	32.6104
	+5	0.4148	2.4198	0.9826	7.8950	1238.3154	16.6962
	+0	0.4050	0	0.9107	0	1061.1442	0
	−5	0.3993	−1.4074	0.8733	−4.1067	913.4203	−13.9212
	−10	0.3770	−6.9136	0.7535	−17.2614	771.0424	−27.3386
a	+10	0.4010	−0.9877	0.8720	−4.2495	1245.0784	17.3336
	+5	0.4033	−0.4198	0.8927	−1.9765	1153.7143	8.7236
	+0	0.4050	0	0.9107	0	1061.1442	0
	−5	0.4064	0.3457	0.9265	1.7349	967.6472	−8.8110
	−10	0.4075	0.6173	0.9406	3.2832	871.7970	−17.8437
b	+10	0.3499	−13.6049	0.9039	−0.7467	1325.2813	24.8917
	+5	0.3970	−1.9753	0.9069	−0.4173	1219.8596	14.9570
	+0	0.4050	0	0.9107	0	1061.1442	0
	−5	0.4202	3.7531	0.9172	0.7137	1002.2714	−5.5480
	−10	0.4961	22.4938	0.9186	0.8675	957.2585	−9.7900
I_p	+10	0.4050	0	0.9107	0	1061.1442	0
	+5	0.4050	0	0.9107	0	1061.1442	0
	+0	0.4050	0	0.9107	0	1061.1442	0
	−5	0.4050	0	0.9107	0	1061.1442	0
	−10	0.4050	0	0.9107	0	1061.1442	0
I_e	+10	0.4145	2.3457	0.9248	1.5483	1145.3904	7.9392
	+5	0.4099	1.2099	0.9179	0.7906	1104.2311	4.0604
	+0	0.4050	0	0.9107	0	1061.1442	0
	−5	0.3998	−1.2840	0.9030	−0.8455	1016.1212	−4.2429
	−10	0.3944	−2.6173	0.8949	−1.7349	969.7853	−8.6095
M	+10	0.4109	1.4568	0.9350	2.6683	1135.7230	7.0281
	+5	0.4080	0.7407	0.9229	1.3396	1098.7579	3.5446
	+0	0.4050	0	0.9107	0	1061.1442	0
	−5	0.4020	−0.7407	0.8982	−1.3726	1023.5109	−3.5465
	−10	0.3988	−1.5309	0.8856	−2.7561	984.5949	−7.2138
N	+10	0.4069	0.4691	0.9234	1.3945	1035.1988	−2.4450
	+5	0.4060	0.2469	0.9171	0.7028	1048.4694	−1.1944
	+0	0.4050	0	0.9107	0	1061.1442	0
	−5	0.4041	−0.2222	0.9042	−0.7137	1074.4838	1.2571
	−10	0.4031	−0.4691	0.8978	−1.4165	1087.2285	2.4581

5.4 Discussion of the Results

It is observed from figure 3 together with tables 1, 2&3 that:

1. As the ordering cost A increases, t_1^*, T^* and TP^* also increase for all cases. In practical situations, when the ordering cost is high, the retailer tends to order more goods, which leads to an increase in the replenishment cycle time and the overall total profit.

2. As the demand rate b increases, t_1^* and T^* decrease while TP^* increases for all cases. This aligns with real-life expectations, as a higher demand rate typically leads to a decrease in the

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optimal replenishment length. In practical situations, when the demand rate is high, retailers tend to order more goods, which results in more sales and an increase in total profit.

3. As the deterioration rate θ increases, t_1^*, T^* and TP^* decrease for all cases. In real-life situations, if the rate of deterioration is high, the replenishment length actually decreases due to the deterioration of goods. This decrease in replenishment length, in turn, leads to a decrease in the total profit.

4. As the selling price p increases, t_1^*, T^* and TP^* also increase for all cases. This aligns with expectations in real life. When the selling price is high and the annual total relevant inventory costs increase, retailers tend to sell fewer items as customers tend to buy from cheaper retailers. Consequently, the time it takes for the inventory to reach zero also increases.

5. As the interest earned I_e increases, t_1^*, T^* and TP^* also increase for all cases. This is also as expected in real life because as the interest earned is increasing the total profit function is also increasing.

6. As the initial demand a is increasing, t_1^* and T^* decrease while TP^* increases for all cases. In a real life situation, a higher demand leads to an increase in sales and hence an increase in total profit.

7. As the interest charged I_p increases, t_1^* and T^* increase while TP^* decreases for all cases. This aligns with real life situation because as the interest payable is increasing the total profit function is decreasing, and the cycle time increases due to delay in payment.

8. As the upstream credit period M increases, t_1^*, T^* and TP^* increase for all cases. In the real world this is expected because an increase in the credit period means an increase in the cycle time, and an increase in the total profit from items sold.

9. As the downstream credit period N increases, t_1^* and T^* increase while TP^* decreases for all cases. In the real world this is expected because an increase in the credit period means an increase in the cycle time, but the total profit decreases due to an increase in interest charged for any delay incurred from N.

10. As the holding cost h and the purchasing cost c increase, t_1^*, T^* and TP^* decrease for all cases. In the real world higher holding cost and purchasing cost leads to higher inventory costs which reduces the total profit. Also the cycle time decreases since the retailer tends to buy fewer items due to high inventory costs.

6 Summary

In this paper, we have studied inventory system for non-instantaneous deteriorating items, which presents challenges in business planning. The existing model by Maihami and Kamalabadi (2011) was expanded to incorporate a demand function that considers both price as well as stock–dependent demand. Additionally, a two–level trade credit system was included to enhance the model's applicability. Various assumptions were made, and a mathematical model with three cases was developed. The non-linearity of the cost functions required the use of the Newton Raphson method to find optimal solutions. For the first case, it was found that $t_1^{1*} = 0.4802$ year (175 days),$T_1^* = 0.9088$ year (332 days), $TP_1^* = 2835.2110$. For the second case, $t_1^{2*} = 0.1218$ year (44 days), $T_2^* = 0.7613$ year (278 days), $TP_2^* = 1249.00173$, and for the third case, $t_1^{3*} = 0.4050$ year (148 days), $T_3^* = 0.9107$ year (332 days), $TP_3^* = 1061.1442$. Sensitivity analyses were carried out on the decision parameters to see how sensitive are they on the model.
In conclusion, based on the numerical examples provided, the analysis demonstrates that the total annual profit is highest in the first case, indicating that it is more cost-effective for the retailer to order larger quantities, extend the inventory depletion period, and adjust the optimal cycle length.

Sensitivity analysis reveals that the holding cost, deterioration cost and the interest charged rate significantly influence the optimal cycle period and the maximum total profit.
It is recommended that the model can be extended to consider stochastic demand function, which would allow for better demand forecasting by incorporating randomness and variability. Furthermore, the model could be extended to include dynamic backlogging so as enhance its practicality and relevance in real-world scenarios.

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