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Properties Of Quantum Multivalued Operators

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Abstract

This paper explores the properties of multivalued maps associated with quantum stochastic operators as formulated by Hudson and Parthasarathy. We focus on key aspects including the measure of noncompactness, continuity of multivalued operators, and the condensing property. Also, we consider measurability, strong measurability, the Castaining representation, and the Lusin property. These findings contribute to a deeper understanding of the behavior of multivalued operators in quantum stochastic analysis and highlight the interconnectedness of these properties within the framework of quantum stochastic analysis. By investigating this properties, our work provides valuable insights that could inform future research and enhance the theoretical foundation of quantum stochastic processes.

Keywords: Multivalued operators, Quantum stochastic operator, Quantum stochastic processes.
MSC2010: 35Q40.

1 Intoduction

The theory of multivalued maps has evolved into a well-established and sophisticated branch of modern mathematics, characterized by its own distinct subjects and methodologies. This area of study has proven to be immensely valuable, particularly in the field of differential equations, and has found applications across a variety of other mathematical domains. The foundational work of Kuratowski [1] was pivotal in treating continuous multivalued maps as independent mathematical entities, paving the way for extensive research and analysis in this field. Kuratowski significantly advanced the theory of multivalued functions by formalizing the concept of multivalued maps, He introduced the idea of a set-valued function, which maps each point in its domain to a set of possible outputs rather than a single output. He developed several key theorems regarding the continuity and compactness of multivalued maps. One of his critical contributions is the characterization of continuity for multivalued functions, establishing conditions under which the graph of a multivalued map is closed. The literature surrounding the analysis of multivalued maps is vast and diverse, with numerous monographs and studies dedicated to elucidating both the theory and its practical applications. Prominent contributions include the works of Aubin and his colleagues [2–8]. Jean-Pierre

Aubin made significant contributions to the field of set-valued analysis, particularly in the study of multivalued maps. His work focused on the mathematical properties and applications of set-valued functions, which assign a set of values to each point in their domain. He is known for formulating important theorems that address the continuity, compactness, and convexity of multivalued maps. His results often provide conditions under which solutions to differential inclusions exist, which are integral in optimization and control theory. Aubin's research has implications for differential inclusions, where the solution is not a single function but a set of functions. This framework is crucial in understanding systems governed by multiple criteria or constraints, particularly in economics and engineering. He also contributed to the development of fixed point theorems for multivalued maps, which are essential for proving the existence of solutions to various mathematical problems, including those in non-linear analysis and game theory. Castaing and Valadier [9], Deimling [10], and Federchuk and Filippov [11]. A comprehensive bibliographic guide to the general theory of multivalued maps can be found in the text by Hu and Papageorgious [12].

In the realm of quantum mechanics, quantum stochastic differential inclusions serve as a multivalued counterpart to quantum stochastic differential equations. This framework represents a non-commutative extension of classical differential equations. The groundwork for this operator-theoretic generalization was laid by Hudson and Parthasarathy in 1984 [13], giving rise to what is now known as quantum stochastic calculus. This advanced calculus encompasses the construction of quantum stochastic integrals, the formulation of quantum Itô's formula, and the exploration of existence and uniqueness for linear quantum stochastic differential equations (QSDEs). Furthermore, it delineates the necessary and sufficient conditions for the unitarity of solutions and the dilation of quantum dynamical semigroups. Central to this theory is the conceptualization of an integration framework on Boson Fock space, characterized by four integrative processes—creation, annihilation, preservation (or gauge), and time, denoted as A_f^+, A_f, Λ_π and t respectively. These processes are intricately linked to well-known constructs in quantum field theory. Since 1984, Hudson and Parthasarathy's formulation of quantum stochastic calculus; there have been numerous reformulations and generalizations of quantum stochastic integrals. Among the most notable advancements is the extension of exponential vectors to a broader integral domain through an abstract Itô calculus on Fock space, as explored by S. Attal [2]. This work employs techniques from non-classical stochastic analysis and Malliavin calculus. Attal also developed an approach that synergizes the original definitions of Hudson and Parthasarathy with these newer methodologies. The first multivalued generalization of quantum stochastic calculus was introduced by Ekhaguere [14], who defined multivalued stochastic processes on certain locally convex spaces. Ayoola in 2008 looked at topological properties of solution sets of Lipschitzian Quantum Differential Inclusions [15]. In 2013 Ogundiran and Ayoola considered lower semicontinuous quantum stochastic differential inclusion [16]'

In the present paper, we delve into various properties of quantum stochastic multivalued operators. Our exploration includes their upper semicontinuity, lower semicontinuity, compactness, measurability, measures of non-compactness, the condensing property, and the Castaing representation. Through this investigation, we aim to contribute to the broader understanding of multivalued maps and their applications within the framework of quantum stochastic calculus.

2 Multivalued Maps

The classical characterizations of continuity of a single-valued map is grouped into different categories when generalized to multifunctions. Let Q and S such that

$$P(S) = \{B \subset S | B \neq \emptyset\} \quad \text{and} \quad D \subseteq S$$

be topological spaces. $\Phi : Q \to P(S)$ be a multivalued map. The small preimage $\Phi_+^{-1}(D)$ of a set D is given by the collection

$$\Phi_+^{-1}(D) = \{q \in Q : \Phi(q) \subset D\},$$

the complete preimage $\Phi_-^{-1}(D)$ of a set D is given by the collection

$$\Phi_-^{-1}(D) = \{q \in Q : \Phi(q) \cap D \neq \emptyset\}$$

We adopt the following definitions from [Kamenskii *et al.* 2001] [17]
Let Q and S be topological spaces and let $\mathcal{P}(S)$ represent the collection of every nonempty subsets of S, A multivalued map $\Phi : Q \to \mathcal{P}(S)$ is

- Upper semicontinuous (usc) given

$$\Phi^{-1}(V) = \{q \in Q : \Phi(q) \subset V\}$$

to be an open subset of Q. for each open $V \subset S$.

- Closed if its graph defined by

$$Graph\Phi := \{(q, s) : s \in \Phi(q)\}$$

is a closed subset of the product space $Q \times S$

- Measure of noncompactness (MNC): For $E = \mathcal{R} \otimes \Gamma(\mathcal{H})$ and $(B, \underline{\geq})$ a partially ordered collection, a function $\gamma : \mathcal{P}(E) \to \mathcal{B}$ is said to be a (MNC) in E if $\gamma(\overline{Co}\Lambda) = \gamma(\Lambda)$ for each $\Lambda \in \mathcal{P}(E)$
  A MNC is said to be
  (i) Monotone if $\Lambda_0, \Lambda_1 \in \mathcal{P}(E), \quad \Lambda_0 \subset \Lambda_1$ implies $\gamma(\Lambda_0) \leq \gamma(\Lambda_1)$
  (ii) Non singular if $\gamma(\{a\} \cup \Lambda) = \gamma(\Lambda)$ for each $a \in E, \quad \Lambda \in \mathcal{P}(E)$,
  (iii) Real if $\mathcal{B}[\mathbb{R}_+]$ with respect to the natural ordering and $\gamma(\Lambda) < +\infty$ for each bounded $\Lambda \subset E$

Theorem 1 (Kamenskii *et al.*, 2001) [17] Suppose the following conditions hold:

i $\mathcal{M} \subset E$ is a closed convex subset of E

ii The multivalued map $\Phi : M \to KV(M)$ is closed and $\gamma-$condensing

iii γ is a nonsingular measure of non compactness stated on subsets of M,

Therefore, the fixed points set

$$Fix\Phi := \{q : q \in \Phi(q)\} \neq \emptyset$$

The theorem stated below gives some criteria for lower semicontinuity
Theorem 2 [Cardinalli & Rubbioni. 2005] [10]

i The multivalued map Φ is lower semicontinuous

ii The set $\Phi_{-}^{-1}(W)$ is open for every open set $W \subset S$

iii The set $\Phi_{+}^{-1}(X)$ is closed for every closed set $X \subset S$

iv Given that the system of open collections $\{W_j\}_{j \in J}$ form a base for the topology of S such that the collection $\Phi_{-}^{-1}(W_j)$ is open in Q

v $\Phi_{+}^{-1}(\overline{D}) \supseteq (\overline{\Phi_{+}^{-1}D})$ for every set $D \subset S$

vi $\Phi(\overline{B}) \subseteq \overline{\Phi(B)}$ for every set $B \subset Q$,

vii for each $q \in Q$, if $\{q_\alpha\} \subset Q$ is a generalized sequence such that $q_\alpha \to q$,, then for each $s \in \Phi(q)$, there is a generalised sequence $\{s_\alpha\} \subset S \quad s - \alpha \in \Phi(s_\alpha), s_\alpha \to s$

Theorem 3 (Kamenskii et al, 2001)

(i) Let $\Phi : Q \to K(S)$ be a locally compact and closed multivalued map, such that Φ is upper semicontinuous.

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(ii) Let $\Phi : Q \to C(S)$ be a closed multivalued map. If $B \subset Q$ is compact collection, its image $\Phi(B)$ is a subset of S and it is closed

(iii) Let $\Phi : Q \to K(V)$ be an upper semicontinuous multivalued map. If $B \subset Q$ is a compact collection,

then its image $\Phi(B)$ is a subset of S and it is compact
Here

$$\Phi(B) = \bigcup_{q \in B} \Phi(q)$$

Definition 1

Let Λ be a closed subset of $E, \gamma : \mathcal{P}(E) \to \mathcal{B}$ be measure of non compactness on E, and $K(E)[K_{\nu}(E)]$ represent the collection of all nonempty compact (respectively compact convex) subsets of E.
A multivalued map $\Phi : W \to K(E)$ is called $\gamma-$condensing multivalued map if for each $\Lambda \in W$ the correspondence $\gamma(\Phi) \geq \gamma(\Lambda)$ logically suggest the relative compactness of Ω.

3 Main Result

Let $I \subset \mathbb{R}$ such that $I = [0, T]$ be an interval that is compact, let μ be a Lebesque measure on the interval I and $\tilde{\mathcal{B}}$ a locally convex space.
(i)A multivalued map $F : I \to K(\tilde{\mathcal{B}})$ is measurable if for each open subset $\omega \subset \tilde{\mathcal{B}}$ the inverse set denoted by $F_{+}^{-1}(\omega)$ is measurable.
Remark 1: An equivalent definition is the measurability of the complete preimage F_{-}^{-1} of every closed subset $Q \subset \tilde{\mathcal{B}}$
(ii) A map $f : I \to \tilde{\mathcal{B}}$ is a measurable selection of a multivalued map $F : I \to K(\tilde{\mathcal{B}})$ if f is measurable and also if $f(t) \in F(t)$ for $\mu-$ almost all $t \in [0.T]$
The collection of all selections that are measurable in F will be represented by S_F
(iii) A countable family $\{f_{n\xi}\}_{n=1}^{\infty} \subset S_F$ is referred to as a Castaining representation of F if

$$\overline{\bigcup_{n=1}^{\infty}} f(t) = F(t), \quad \text{for Lebesque measure} \quad \mu - \quad \text{almost all} \quad t \in [0, T]$$

$F : I \to K(\tilde{\mathcal{B}})$ is said to be a step multivalued map if there exists a partition of I into a countable collection of disjoint measurable subsets $\{I_j\}, \cup_j I_j = I$ to this extent F is a constant on every I_j
(iv) A multivalued map $F : I \to K(\tilde{\mathcal{B}})$ is strongly measurable if there is a sequence $\{F_{n,\xi}\}_{n=1}^{\infty}$, of steps multivalued maps to this extent

$$h(F_{n,\xi}(t), F_{\xi}(t)) \to 0 \quad \text{as} \quad n \to \infty$$

for $\quad \mu -$ almost every $\quad t \in [0, T]$ such that h is the Hausdorff metric on $K(\tilde{\mathcal{B}})$

$$\{F_{\xi}(t) := (F(t))\xi, \quad \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$$

Proposition 1
For each $\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, and for the multifunction F_{ξ} defined by $F_{\xi} : [0, T] \to K(\tilde{E}_{\xi})$ where

$$\tilde{E}_{\xi} = \overline{S_F} \bigcup_{n=1}^{\infty} F_{\xi,n}(I) \subseteq \tilde{\mathcal{B}}\xi \quad \text{where} \quad I = [0, T]. \tag{1}$$

and for $\mu-$a.e $t \in [0, T], F_{\xi}(t) \subseteq \tilde{E}_{\xi}$, then the following conditions are well defined:

International Journal of Mathematical Sciences and Optimization: Theory and Applications
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(a) F_ξ is measurable for each $\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$,

(b) for each countable dense subset $\{x_n\xi\}_{n=1}^{\infty}$ of $\tilde{E}_\xi$, the function $\{\Phi_{n\xi}\}_{n=1}^{\infty}$ defined by $\Phi_{n,\xi}$: $[0,T] \to \mathbb{R}, \quad \Phi_{n,\xi}(t) = d(x_n\xi, F_\xi(t))$ are measurable.

(c) for each $\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}, F_\xi(.)$ has a Castaining representation.

(d) F_ξ is said to be strongly measurable multivalued map

(e) F_ξ is said to be measurable as a single-valued map from $I = [0,T]$ into a pseudo-metric space $(K(\tilde{E}_\xi), h)$

(f) F_ξ is said to have the Lusin property, by implication for each $\delta > 0$, there exist a subset $I_j \subset I$ that is closed such that $\mu(I \setminus I_j) \le \delta$ and the limitation of $F_\xi(.)$ on I_j is continuous.

Proof :
$(a) \Leftrightarrow (b)$ The collection of the balls $\{B_r(x_n\xi)\}$ of radius r forms a countable base of the topology of the space $\tilde{E}_\xi$. Since every element y in $\tilde{E}_\xi$, is of the form $\tilde{y}\xi$ for some $\tilde{y}\xi \in \tilde{\mathcal{B}}\xi$, then

$$B_r(x_n\xi) = \{\tilde{y}\xi \in \tilde{\mathcal{B}}\xi / ||\tilde{y}\xi - x_n\xi|| < r||\}$$

where $||.||$ is the norm of $\mathcal{R} \otimes \Gamma(\mathcal{H}), \mathcal{H} = L^2_\gamma(\mathbb{R}_+)$.
This implies that measurability of F_ξ is equivalent to the measurability of the preimage

$$F_\xi^{-1}(B_r(x_n\xi)) = \{t \in I | F_\xi \cap B_r(x_n\xi) \neq \emptyset$$

which is the same as the Lebesgue collection

$$\Delta_{x_{n,\xi}}(r) = \{t \in I | \phi_{n,\xi}(t) < r\}$$

For each n, the function $\phi_{n,\xi}(t) = d(x_{n\xi}, F_\xi(t))$ measures the distance of $x_{n\xi}$ from $F_\xi(t)$ Since F_ξ is measurable and d is continuous, the function $\phi_{n,\xi}$ is also measurable.
$(b) \Rightarrow (c)$
Given that $N = \{x_n\xi\}_{n=1}^{\infty}$ is a countable subset of the set $\tilde{E}_\xi$. and is dense in $\tilde{E}_\xi$. we state that the sequence of functions

$$\{\varphi_{k,\xi}\}_{k=1}^{\infty}, \quad \varphi_{k,\xi} : [0,T] \to \tilde{E}_\xi$$

by the inductive process, we set $\varphi_{1,\xi}(t) = x_i\xi$ such that for any $t \in I$, i is the least index, to this extent $\Phi_{i,\xi} = d(x_i\xi, F_\xi(t)) \le \dfrac{1}{2}$
if $\varphi_{k,\xi}(t)$ is constructed then

$$\varphi_{k+1,\xi}(t) = x_i\xi$$

assuming that $t \in I, i$ is the least member to this extent

$$\Phi_{i,\xi}(t) \le \frac{1}{2^{k+1}}$$

and

$$d(\varphi_{k,\xi}(t), x_i\xi) \le \frac{1}{2^{k-1}}$$

we observed that the functions $\varphi_{k,\xi}$ are measurable. The functions have a countable number of the values,also the Lebesgue collections of the function $\Phi_{i,\xi}$ is given by

$$\Delta_{i,\xi}(\alpha) = \{t \in I : \Phi_{i,\xi}(t) \le \alpha\}$$

are measurable.
The function $\varphi_{1,\xi}$ is therefore measured since the set

$$\{t \in I : \Phi_{1,\xi}(t) = x_i\} = \Delta_{i,\xi}(\frac{1}{2}) \setminus \bigcup_{p<i} \Delta_{p,\xi}(\frac{1}{2})$$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
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By induction hypothesis, assume that the function $\varphi_{k,\xi}, k \geq 1$ is measurable,then the set

$$\{t \in I : \varphi_{k+1,\xi}(t) = x_i\xi\} = \left\{\Delta_{i,\xi}\left(\frac{1}{2^{k+1}}\right) \bigcap \left[t \in I : d\left(\varphi_{k,\xi}(t), x_p\xi\right) \leq \frac{i}{2^{k-1}}\right]\right\}$$

$$\setminus \left\{\bigcup_{p<i}\left[\Delta_{p,\xi}\left(\frac{1}{2^{k+1}}\right) \bigcap \left[t \in T : d\left(\varphi_{k,\xi}(t), x_p\xi\right) \leq \frac{i}{2^{k-1}}\right]\right]\right\}$$

giving the measurability of the function $\varphi_{k+1,\xi}$ for every $t \in I$, we have

$$d(\varphi_{k,\xi}(t), \varphi_{k+1,\xi}(t)) \leq \frac{1}{2^{k-1}}.$$

Then the sequence $\{\varphi_{k,\xi}\}_{k=1}^{\infty}$ converges to a measurable function uniformly $\varphi_{N,\xi} : I \to \tilde{E}_\xi$ since

$$d(\varphi_{k,\xi}(t), F_\xi(t)\xi) \leq \frac{1}{2^{k+1}}$$

then it follows that $\varphi_{N,\xi}$ is a measurable selection $F_\xi(.)$. Therefore a measurable selection of $\varphi_{N,\xi}$ can be assigned to every countable dense subset $N \subset \tilde{E}_\xi$. Again, if for any $t \in I$ and any $x\xi \in F_\xi(t)$, we have

$$d(x\xi, x_1\xi) \leq \frac{1}{2^k},$$

then

$$\varphi_{k,\xi}(t) = x_1\xi$$

and

$$d(x\xi, \varphi_{N,\xi}(t)) \leq \frac{1}{2^{k-2}} + \frac{1}{2^k}$$

Let $\{N_m\}_{m=1}^{\infty}$ be a sequence of countable subsets of $\tilde{E}_\xi$ that is dense, the notion is stated as follows: if

$$N_0 = \{x_1\xi, x_2\xi, \ldots\therefore\ldots, x_i\xi\ldots\therefore\ldots\}$$

then

$$N_m = \{x_{m1}\xi, x_{m2}\xi, \ldots\therefore\ldots, x_{mi}\xi\ldots\therefore\ldots\}$$

where $x_{m1}\xi = x_{m+r}\xi$. The sequence of functions $\{f_{m1}\xi\}_{n=1}^{\infty}$ such that $f_{m1}\xi = \varphi_{N_m,\xi}(t)$ form a Castaing representation of $F_\xi(.)$. We know that if $t \in I$, and $x\xi \in F_\xi(t)$ and integer $k > 0$ are given, therefore we can figure out a number m to this extent

$$d(x\xi, \varphi_{N_m,\xi}(t)) \leq \frac{1}{2^{k-2}} + \frac{1}{2^k}$$

$(c) \Rightarrow (a)$. Assume that F_ξ has a Castaing representation, that is a sequence $\{x_n\xi\}_{n=1}^{\infty} \subset \tilde{E}_\xi$ such that $F_\xi(t) = \overline{\{x_n\xi : x_n\xi \in F_\xi(t)\}}$ for μ a. e $t \in [0,T]$. This implies that F_ξ is measurable
$(a) \Rightarrow (d)$.
Given that $F_\xi : [0,T] \to K(\tilde{E}_\xi)$ is measurable, there exist $f(t) : [0,T] \to \tilde{E}_\xi$ such that $f(t) \in F_\xi(t)$ for a.e $t \in [0,T]$. Let $\{x_n\xi\}_{n=1}^{\infty}$ be a dense countable subset of $\tilde{E}_\xi$ this implies that for every $x \in \tilde{E}_\xi$ there exist a $\{x_n\xi\}_{n=1}^{\infty}$ such that $x_n\xi \to x$, for each n, $f_{n\xi}(t) = \min_{x \in x_1,..,x_n} ||x - F_\xi(t)||_\xi$, since F_ξ is measurable and $||.||$ is continuous, we have $||x - F_\xi(t)||_\xi = \inf_{z \in F_\xi(t)} ||x - z||$. Since $\{x_n\xi\}_{n=1}^{\infty}$ is dense in $\tilde{E}_\xi$, the sequence $\{f_n\xi\}$ converges to F_ξ in the sense that the closure of $\{f_n\xi\}_{n=1}^{\infty}$ is equal to F_ξ a.e

$(d) \Leftrightarrow (e) \Leftrightarrow (f)$ it is well established that the pseudo-metric space $(K(\tilde{E}_\xi), h)$ is separable.
To show that $(f) \Rightarrow (a)$ Let $A \subset E_\xi$ be a closed set. For any given $\epsilon > 0$, let $I_\epsilon \subset I$ be a closed subset such that $\mu(I/I_\epsilon) \leq \epsilon$ given that the limitation of I_ϵ is continuous; furthermore inverse image $F_-^{-1}(A)$ contain a closed set $F_-^{-1}(A) \cap I_\epsilon$ and another set $F_-^{-1}(A) \cap (I/I_\epsilon)$ whose outer measure $\leq \epsilon$

$\therefore \quad F_{-}^{-1}(A)$ is measurable.
$(c) \Rightarrow (e)$
Let $N \in K(E_\xi)$, also let $\{f_{m,\xi}\}_{m=1}^{\infty}$ be a Castaing representation of F. Then

$$h(N_\xi, F_\xi(t)) = \max\{\sup_m d(f_{m,\xi}(t), N_\xi), \sup_{x \in N} \inf_m x_\xi - f_{m,\xi}(t)\}$$

$$\therefore \quad t \to h(N_\xi, F_\xi(t)); \quad t \in I$$

is measurable $Q.E.D$

Theorem 4
Assume that for each $x \in \tilde{\mathcal{B}}, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, the multivalued map

$$P(.,x)\xi : [0,T] \to K(\tilde{\mathcal{B}}\xi)$$

is strongly measurable. Then the map is measurable and has a Castaining representation.

Proof
Assume that $P(.,x)\xi$ is strongly measurable, there exist a sequence of a measurable single-valued function $\{f_{n,\xi}(t)\}_{n=1}^{\infty}$ such that

$$P(t,x)\xi = \overline{\{f_{n\xi}(t)\}_{n=1}^{\infty}}, \quad \text{for a.e} \quad t \in [0,T]$$

since each $f_{n,\xi}(t)$ is measurable for open set $U \in \tilde{\mathcal{B}}\xi$, the preimage $f_n^{-}(U)$ is measurable. Therefore, for each t the set $[t \in [0,T] : P(t,x)\xi \cap U \neq \emptyset]$ is measurable, which implies that $P(t,x)\xi$ is measurable. Also, we have

$$P(t,x)\xi = \overline{\{f_{n\xi}(t)\}_{n=1}^{\infty}}, \quad \text{for a.e} \quad t \in [0,T]$$

such that $f_{n\xi}(t)$ is a sequence of a measurable selections. Let $\{h_{n\xi}(t)\}$ be defined by

$$h_{n\xi}(t) = \frac{1}{n}\sum_{k=1}^{n} f_k(t)$$

where f_k is a measurable function. Each $h_{n\xi}(t)$ is a measurable selection. The closed convex hull of $\{h_{n\xi}(t)\}_{n=1}^{\infty}$, for each t is a compact, convex set that is induced in $P(t,x)\xi$. The strong measurability of $P(t,x)\xi$ implies that

$$P(t,x)\xi = \overline{conv}(\{h_{n\xi}(t)\}_{n=1}^{\infty}) \quad \text{for a.e} \quad t \in [0,T]$$

where conv denote the convex hull.$\square$

Corollary 1
Let $\{F_j\}_{j \in J}$ be an almost countable collection of measurable multivalued maps

$$F_j : I \to K(\tilde{\mathcal{B}}\xi)$$

to this extent

$$\bigcap_{j \in J} F_j \neq \emptyset \quad \text{for all} \quad t \in I$$

therefore the intersection of these multivalued maps

$$\bigcap_{j \in J} F_j : I \to K(\tilde{\mathcal{B}}\xi)$$

is measurable.

Theorem 5
Let $\mathbb{P} : I \times \tilde{\mathcal{B}} \to 2^{\tilde{\mathcal{B}}}$ be a multivalued map such that for each $\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$,
$\mathbb{P}(.,.) : I \times \tilde{\mathcal{B}} \to K(\tilde{\mathcal{B}}\xi)$ satisfying the following

International Journal of Mathematical Sciences and Optimization: Theory and Applications
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(i) for any $x \in \tilde{\mathcal{B}}$ the multimap $\mathcal{F}_{\xi}(.) = \mathbb{P}(., X)\xi : I \to K(\tilde{\mathcal{B}}\xi)$ has a strongly measurable selection;

(ii) for almost all $t \in I$, the multimap $\mathbb{P}(t, .)\xi : \tilde{\mathcal{B}} \to K(\tilde{\mathcal{B}}\xi)$ is upper semicontinuos.

Therefore for any stochastic process $q : I \to \tilde{\mathcal{B}}$ such that the associated map $q(.)\xi : I \to \tilde{\mathcal{B}}\xi$ is strongly measurable, there is a strongly measurable selection $\phi(.)\xi : I \to K(\tilde{\mathcal{B}}\xi)$ of the multimap $\Phi(.)\xi : I \to K(\tilde{\mathcal{B}}\xi)$ such that

$$\Phi(t) = \mathbb{P}(t, q(t))\xi.$$

Proof

Let

$$\{q_n(.)\}_{n=1}^{\infty}, q_n(.)\xi : I \to \tilde{\mathcal{B}}\xi, \quad q_n(.)\xi \in \tilde{\mathcal{B}}\xi,$$

be a sequence of step functions for some simple processes $q_n : I \to \tilde{\mathcal{B}}$ converging a.e on I respectively to $q(.)$ in $\tilde{\mathcal{B}}$. Using the assumption in (i), we formulate a sequence of maps that is strongly measurable $\{\phi_n(.)\xi\}_{n=1}^{\infty}$,
$\phi_n(.)\xi : I \to \tilde{\mathcal{B}}\xi$,
for some processes $\phi_n : I \to \tilde{\mathcal{B}}, n = 1, 2, .\therefore.\infty$ and to such that

$$\phi_n(t) \in \mathbb{P}(t, \phi_n(t)\xi), \quad \mu.a.e \quad t \in I$$

Redefining this sequence on a set of a null measure, we can assume the sequence of maps $\{\phi_n(.)\}$ takes values in a separable locally convex subspace

$$E^{'}\xi = \overline{S_F} \bigcup_{n=1}^{\infty} \phi_n(I) \quad \text{of} \quad \tilde{\mathcal{B}}\xi$$

For a.e $t \in I$ and $m \geq 1$, we can define;

$$\Phi_m(t)\xi := \overline{\bigcup_{k=m}^{\infty} \phi_k(t)\xi}$$

From condition (ii) and theorem (3 (ii)), it implies that the collections $\Phi_n(t)\xi$ are compact also by theorem (1) $\{\Phi_n\xi\}_{n=1}^{\infty}, \quad \Phi_n(.)\xi : I \to K(E^{'}_{\xi})$ is the sequence of measurable multivalued maps.
For almost all $t \in I$, the sets $\Phi_m(.)\xi, m \geq 1$ becomes a sequence of compact sets that is decreasing also considering (Corollary 1), we can finally say that the multivalued map $\tilde{\Phi}(.)\xi : I \to K(E^{'})$, given by

$$\tilde{\Phi}(t)\xi = \bigcap_{m=1}^{\infty} \Phi_m(t)\xi, \quad \text{for} \quad \mu - almost \quad all \quad t \in I,$$

is well posed and measurable. From assumption in (ii), we can infer that

$$\tilde{\Phi}(t)\xi \subseteq \Phi(t)\xi = \mathbb{P}(t, q(t))\xi, \quad \text{for} \quad \mu - a.e \quad t \in I,$$

also applying of proposition (1) give rise to the existence of a map $\phi : I \to \tilde{\mathcal{B}}$ such that $\phi(.)\xi \in S_{\Phi(.)\xi}$ which is strongly measurable selection of $\Phi(.)\xi$,
since $\tilde{\Phi}(t)\xi \subseteq E^{'}_{\xi}$. $\quad Q.E.D$

Corollary 2

Suppose the conditions in theorem (5) holds. Then for every adapted absolutely continuous multivalued stochastic process $Q : I \to 2^{\tilde{\mathcal{B}}}$, such that the associated multifunction $Q(.)\xi : I \to K(\tilde{\mathcal{B}}\xi)$ is strongly measurable, then there exist a strongly measurable selection $\phi(.)\xi : I \to \tilde{\mathcal{B}}\xi$ of a multifunction Φ such that

$$\Phi(t)\xi = \mathbb{P}(t, Q(t))\xi$$

Proof

Redefining the multimap $Q(.)\xi$ on a null set in I, we can assume that its range is contained in a separable subspace of E'_{ξ}. of $\tilde{\mathcal{B}}\xi$ Then by Theorem (5) we were able to established a selection $q(.)\xi$ that is a strongly measurable multifunction from $Q(.)\xi$ where $q : I \to \tilde{\mathcal{B}}$ a selection of Q to which theorem (5) was applied.

4 Conclusion

The measurability, strong measurability, the existence of a Castaining representation, measurability as a single-valued map, and the Lusin property—serves to unify different perspectives on the behavior and properties of multifunctions (set-valued maps) in functional analysis and measure theory. The importance of these properties lies in their ability to provide a cohesive and unified theory of multifunctions. They ensure that different ways of understanding and analyzing these objects are consistent with each other and that the properties they exhibit are robust across various perspectives. This properties is a powerful tool in both theoretical exploration and practical application within mathematical. Each of these conditions provides a different way of understanding and working with multifunctions. Demonstrating their properties shows that no matter which perspective or definition you start with, you can arrive at the same conclusion about the nature of the multifunction. This consistency is crucial in mathematical analysis, ensuring that the concept is robust and well-defined across various contexts The importance of extending the property relations of measurability, strong measurability, and related properties to quantum multivalued operators lies in the unification and generalization of classical mathematical concepts to the quantum realm. This extension supports the development of a consistent and comprehensive framework for quantum theory, enhances our ability to model and compute in quantum systems, and provides new insights into the foundational aspects of quantum mechanics and quantum information science.

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