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# Stable Quasi-Idempotents of Higher Defects in Transformation Semigroups

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**Abstract**

This article investigates the structure of stable quasi-idempotents $\xi$ of arbitrary defect $d(\xi) \geq 1$. We show that, unlike transpositions, stable quasi-idempotents generate the singular transformation semigroups $T_n \setminus S_n$ and $P_n \setminus S_n$, with the inclusion $T_n \setminus S_n \subseteq P_n \setminus S_n$. These semigroups are significant because every finite semigroup is either a subsemigroup or an embedding of them, highlighting the universality of $P_n$. We classify stable quasi-idempotents in terms of their defects and path-cycle structures, establishing explicit enumerative formulas. In particular, a defect-1 stable quasi-idempotent of span $s$ has rank

$$\binom{n}{s} = \frac{n!}{(n-s)!s!}.$$

This classification clarifies the relationship between stable quasi-idempotents, idempotents, and quasi-idempotents, and provides a framework for analyzing the subsemigroups they generate. Our results connect classical work on transformation semigroups with new enumerative and structural insights.

**Keywords:** Rank, Local depth, Global depth, Transformation semigroup.
**MSC2010: 20M20.**

## 1 Introduction

Howie [1] bridged the gap between the work of [2] and that of [3]. [3] introduced the notion of defect $d(\alpha) = |X_n \setminus X_n\alpha|$, collapse $\text{coll}(\alpha) = \{t\alpha^{-1} : t \in X_n\alpha, |t\alpha^{-1}| \geq 2\}$, and shift $\text{shift}(\alpha) = \{x \in X_n : x\alpha \neq x\}$, which are respectively how far an element $\alpha \in T_n$ is from being onto (surjection), how far an element is from being one-to-one (injection), and how far an element is from being both one-to-one and onto (bijection).

[3] used regularity and idempotent generation in establishing Cayley-Suschkewitsch's Theorem: "every finite semigroup is embeddable in a finite full transformation semigroup." Since the inverse element in the symmetric group $S_n$ satisfies the regularity condition $axa = a \in S_n$, whence $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$, and the identity element $e \in S_n$ satisfies the idempotent condition $e^2 = e$, and $S_n$ is a semigroup (being a group), regularity and idempotents are required in embedding every finite semigroup in the finite full transformation semigroup $T_n$, as used by [3].

[1] used the definitions of defect, collapse, and shift to extend the embeddability to infinite semigroups: "every infinite semigroup is embeddable in the infinite full transformation semigroup." The reason is simple: $T_1$ is regular and idempotent generated, and whenever $T_n$ is regular and idempotent generated, then $T_{n+1}$ is also regular and idempotent generated; thus, $T_\infty$ is regular and idempotent generated. However, $T_N$ (where $N \propto 1/\epsilon$ and $0 < \epsilon < 1$) could be used for $T_\infty$, but the formal axiomatic system and algebraic structure of $\infty$ is the aleph naught or continuum hypothesis.

The notion of singular transformations entered the study of semigroups of mappings through axiomatic investigations. In particular, [4] introduced the term in connection with matrix representations rather than working directly with the abstract algebraic structure, while [5] developed the semigroup-theoretic approach to the same objects. The initial formulation of [1] was groundbreaking and has influenced subsequent research for over three decades. Furthermore, [3] emphasized that the symmetric group $S_n$ is naturally embedded in the full transformation semigroup $T_n$. Hence, in the study of embeddability, if every finite semigroup embeds into some $T_n$, then every finite group also embeds into some $T_n$, since $S_n \subseteq T_n$.

Since the idempotent notation $\binom{i}{j}$, standing for $\{i, j\} \to \{j\}$, generates only $T_n \setminus S_n$, then either the notation needs refinement or the choice of the $m$-potent should be replaced. The symmetric group $S_n$ is generated by products of transpositions, which are cycles of length 2. Digraphically, an idempotent does not contain a cycle of length 2, but a quasi-idempotent does. Thus, the quasi-idempotent $\delta = \delta^2 = \delta^4$ (as adopted by [6] and recently adopted by [7]) generates $T_n$, not just $T_n \setminus S_n$.

By the definition of stabilizer $s$ [8], we have $\operatorname{im}\alpha^s = \operatorname{im}\alpha^{s+1}$; the quasi-idempotent is the only quasi-idempotent that is stable. The subset containing the stable quasi-idempotents SQI of the subsemigroups $T_n \setminus S_n$ or $P_n \setminus S_n$ [through Vagner's Theorem [9]] is our underlying set having a rank:

$$\operatorname{rank}(T_n \setminus S_n) = \{|\mathrm{SQI}| : \mathrm{SQI} \subseteq T_n \setminus S_n, \langle \mathrm{SQI} \rangle = T_n \setminus S_n\}.$$

[10] used the notation $\gamma_y^x(z) = \begin{cases} y \to z = x \\ z \to z \neq x \end{cases}$ for an idempotent of defect 1 and derived the length of gravity:

$$g(\gamma_y^x(z)) = n + c(\gamma_y^x(z)) - f(\gamma_y^x(z)),$$

where $c(\gamma_y^x(z))$ is the number of cycles of $\gamma_y^x(z) \in T_n$ and $f(\gamma_y^x(z))$ is the number of acyclic and trivial orbits of $\gamma_y^x(z) \in T_n$. Unlike the force of gravity, Iwahori's derivation of the length of gravity was difficult to comprehend, but with Howie's refinement (1980), it became simple and elegant, akin to the force of gravity due to Newton. The length of gravity is attributed to both Iwahori and Howie today.

The study of quasi-idempotents within finite transformation semigroups continues to reveal deep structural insights into the interplay between algebraic and graph-theoretic representations of transformations. Recent investigations, such as [11], have established a universal classification of elements in the finite partial transformation semigroup $P_n$ acting on the set $X_{n+1} = \{0, 1, 2, \ldots, n\}$. Their work emphasizes the relationship between the powers of transformations, equivalence relations, and the cyclic as well as quasi-idempotent structures of these transformations. A key observation is that for any transformation $\alpha \in P_n$, repeated application leads to a stable, periodic configuration that can be described by an $(m, r)$-path cycle—capturing both cyclic and linear behaviors within the orbit structure of $\alpha$.

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In this framework, orbits are treated as equivalence classes under the relation $x \sim y$ if $x\alpha^m = y\alpha^r$, providing a coherent description of the dynamics of $\alpha$. These orbit structures form the basis for distinguishing idempotent elements ($\varepsilon^2 = \varepsilon$) from quasi-idempotent elements ($\xi^2 \neq \xi$, yet $\xi^4 = \xi^2$). Notably, stable quasi-idempotents play a central role in generating the ideal $P_n \setminus S_n$, where $S_n$ denotes the symmetric group, showing that quasi-idempotent elements constitute the connective tissue between the permutation and non-permutation parts of $P_n$. This digraphic decomposition offers a visual and algebraic understanding of the semigroup's internal structure and its ideal hierarchy.

Complementary results by [12] extend this investigation to the full order-preserving transformation semigroup $O_n$, presenting both algebraic and graph-theoretic characterizations of stable quasi-idempotents. They establish that a transformation $\xi \in O_n$ is a stable quasi-idempotent if and only if it satisfies $\xi \neq \xi^2$ and $\xi^2 = \xi^3$. Graph-theoretically, the associated functional digraph $G_\xi$ of such a transformation possesses at least one fixed point, no cycles of length greater than one, and ensures that every vertex either is a fixed point or maps to a fixed point within at most two steps. These conditions guarantee stability and structural predictability within $O_n$. The explicit enumeration of functional digraphs for $n = 4$ and the cubic polynomial formula for small $n$ values,

$$a_n = \tfrac{43}{3}n^3 - 157n^2 + \tfrac{1736}{3}n - 708,$$

highlight the combinatorial richness of stable quasi-idempotents and the computational role of the GAP software in validating these theoretical predictions.

Building upon these foundational works, we consider the algebraic implications of stability conditions in quasi-idempotents of defect 1. Using the notation introduced in [11], a stable quasi-idempotent of defect 1 is defined as

$$\gamma_y^x(w) = \begin{cases} y, & \text{if } w = x, \\ z, & \text{if } w = y, \\ w, & \text{if } w \neq x \neq y, \end{cases}$$

and the *length of gravity* with respect to this element follows from the relation

$$\gamma_y^x(z)\,\gamma_z^y(w) = \gamma_y^x(w).$$

Historically, Howie (1966–1978) established the rank of the full transformation semigroup $T_n$ as

$$\operatorname{rank}(T_n) = \frac{n(n-1)}{2},$$

after twelve years of research, and subsequently derived the local and global depths respectively given by

$$g(\alpha) = n + c(\alpha) - f(\alpha) \quad \text{and} \quad G(T_n) = \tfrac{3}{2}(n-1).$$

Garba [13] later proved that $g(\alpha) = g(\alpha^*)$ since $f(\alpha^*) = f(\alpha) + 1$, where $\alpha^* \in P_n^*$, a subsemigroup of $T_{n+1}$ that is isomorphic to $P_n$ via Vagner's Theorem. Furthermore, Garba [14] established that the rank of $P_n$ is

$$\operatorname{rank}(P_n) = \frac{n(n+1)}{2},$$

obtained by substituting $n+1$ for $n$ in Howie's formula. This holds because $P_n$ lies between $T_n$ and $T_{n+1}$, satisfying

$$T_n \hookrightarrow P_n \hookrightarrow T_{n+1}.$$

Since both the local depth $g(\alpha)$ and the ranks of $T_n$ and $P_n$ (that is, $\frac{n(n-1)}{2}$ and $\frac{n(n+1)}{2}$, respectively) are related, it follows that the global depth of $P_n$ does not exceed $\frac{3n}{2}$, obtained by replacing $n-1$ with $n$ in $\frac{3}{2}(n-1)$.

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A stable quasi-idempotent of defect 1 is either a *3-path* or a *2-chain*. The total number of 3-paths and/or 2-chains in $P_n$ is $n(n-1)^2$, whereas $T_n$ has a stable quasi-idempotent rank of

$$3! \cdot \binom{n}{3} = n(n-1)(n-2),$$

and $T_{n+1}$ has a stable quasi-idempotent rank of

$$3! \cdot \binom{n+1}{3} = (n+1)n(n-1).$$

Thus, the rank of $P_n$ lies between those of $T_n$ and $T_{n+1}$ since

$$n(n-1)(n-2) < n(n-1)^2 < (n+1)n(n-1).$$

These relationships reinforce the structural consistency of partial and order-preserving transformation semigroups, demonstrating that their quasi-idempotent elements encode not only algebraic regularities but also deep combinatorial symmetries across nested semigroup classes.

## 2 Preliminaries

Since [1] considered defect 1 idempotent generating elements, [15] extended it to higher defects $d(\alpha)$ and found the local depth $k(\alpha) = \left\lceil \frac{g(\alpha)}{d(\alpha)} \right\rceil$ or $k(\alpha) = \left\lceil \frac{g(\alpha)}{d(\alpha)} \right\rceil + 1$. [16] used products of 3-paths to obtain the depth as $\frac{1}{2}\lceil g(\alpha) + m(\alpha) \rceil$, where $m(\alpha)$ is the measure of $\alpha$. Since stable quasi-idempotents are 3-paths in $T_n$ and the set of stable quasi-idempotents is a subset of the set of 3-paths in $P_n$, stable quasi-idempotents of arbitrary defect will also satisfy

$$\frac{1}{d(\xi)} \lceil g(\xi) + m(\xi) \rceil \leq \frac{2g(\xi)}{d(\xi)}.$$

Thus, we have the local depth as

$$\frac{1}{d(\xi)} \lceil g(\xi) + m(\xi) \rceil$$

and the rank as

$$\frac{2g(\xi)}{d(\xi)}.$$

**Definition 2.1.** *[3]*A mapping $\alpha : X_n \to X_n$ is called a full (or total) transformation of $X_n$. The set $T_n$ of all full transformations of $X_n$ forms a semigroup under the composition of mappings called the full transformation semigroup.

**Definition 2.2.** *[17]*The element $\alpha \in P_n$ is called an idempotent if $\alpha = \alpha^2$ [1]. The element $\beta \in P_n$ is said to be a quasi-idempotent if $\beta \neq \beta^2 = \beta^4$ [6]. The element $\xi \in P_n$ is said to be a stable quasi-idempotent if $\xi \neq \xi^2 = \xi^3$.

**Definition 2.3.** *[18]*The index and period of an element $a$ of a finite semigroup are the smallest values of $m \geq 1$ and $r \geq 1$ such that $a^{(m+r)} = a^m$. An element with index $m$ and period 1 is called an $m$-potent element. Let $S$ be a semigroup and $a \in S$. If there exist $m, r \in \mathbb{Z}^+$ such that $a^{(m+r)} = a^m$ and $a, a^2, \ldots, a^{(m+r-1)}$ are pairwise distinct, then $a$ is called an $(m, r)$-potent element of $S$, and we say that $a$ has index $m$ and period $r$. In particular, if $r = 1$, then $a$ is called an $m$-potent, and if $m = r = 1$, then $a$ is called an idempotent.

**Definition 2.4.** *[3]Let* $X_n = \{1, 2, \ldots, n\}$ and let $T_n$ be the full transformation semigroup on $X_n$. If $\{x_1, \ldots, x_m\} \subseteq X_n$ and $\alpha \in T_n$ for some $1 \le r \le m$ such that

$$x\alpha = \begin{cases} x_{i+1} & \text{if } x = x_i \quad (1 \le i \le m-1) \\ x_r & \text{if } x = x_m \\ x & \text{if } x \in X_n \setminus \{x_1, x_2, \ldots, x_m\} \end{cases}$$

then $\alpha$ is called a path-cycle of length $m$ and period $r$, or simply, an $(m, r)$-path cycle, and is denoted (in linear notation) by $\alpha = [x_1, x_2, \ldots, x_m | x_r]$.

An $(m, r)$-path cycle $\alpha$ is called:

- *$m$-path* if $r = m$;
- *$m$-cycle* if $m \ge 2$ and $r = 1$;
- *a loop* if $m = r = 1$.
- *an idempotent of defect 1* if $m = r = 2$
- *a proper path-cycle* if $m \ge 2$ and $r \ne 1$.

Let $\xi = [x_1, x_2, x_3 | x_3]$ be an arbitrary 3-path in $\text{Sing}_n$, then $\xi$ maps $x_1$ to $x_2$, $x_2$ to $x_3$, and all other elements of $X_n$ identically. Thus, $\xi = \begin{pmatrix} x_2 & x_1 \\ x_3 & x_2 \end{pmatrix}$ in array notation.

**Definition 2.5.** ((m,r)-Path Cycle in $P_n$) [14]. For $1 \le r < m < n$, an $(m, r)$-path-cycle $[x_1, \ldots, x_m | x_r]$ in $P_n^*$ corresponds to an $(m, r)$-path-cycle $[x_1, \ldots, x_m | x_r]$ in $P_n$, while an $m$-path $[x_1, \ldots, x_m | x_m]$ in $P_n^*$ corresponds either to an $m$-path $[x_1, \ldots, x_m | x_m]$ in $P_n$ if $x_m \ne 0$, or to an $(m-1)$-chain $[x_1, \ldots, x_{m-1}]$ in $P_n$ if $x_m = 0$, where $P_n^*$ is the subsemigroup of $T_{n+1}$ that is isomorphic to $P_n$.

**Definition 2.6.** [19] (Shift-Collapse and Defect) For any $\alpha \in T(X)$, we define the sets $S(\alpha)$, $Z(\alpha)$, and $C(\alpha)$ by

$$S(\alpha) = \{x \in X : x\alpha \ne x\}, \quad Z(\alpha) = X \setminus X\alpha, \quad C(\alpha) = \bigcup \{t\alpha^{-1} : t \in X\alpha, |t\alpha^{-1}| \ge 2\},$$

and refer to the cardinals $|S(\alpha)|$, $|Z(\alpha)|$, and $|C(\alpha)|$ respectively as the shift, the defect, and the collapse of $\alpha$. For each $\alpha \in T_n$, define;

$$\text{Fix}(\alpha) = \{x \in X_n : x\alpha = x\},$$

and denote $X_n \setminus \text{Fix}(\alpha)$ by $\text{Shift}(\alpha)$.

**Definition 2.7.** (Araujo & Konieczny, 2013) (Stabilizer).
The stabilizer $s$ is a positive integer in $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$ such that $\text{im}(\alpha^s) = \text{im}(\alpha^{s+1})$, while the stabilizing image of $\alpha \in T$, denoted by $\text{sim}(\alpha)$, is the set of elements $s \in S$ such that $s$ is a stabilizer.

**Definition 2.8** (Defect). *[14]Let* $\alpha \in T_n$ (or $P_n$) be a transformation on a finite set $X$ with $|X| = n$. The *defect* of $\alpha$, denoted $d(\alpha)$, is defined as

$$d(\alpha) = n - |\operatorname{Im}(\alpha)|,$$

where $\operatorname{Im}(\alpha)$ is the image of $\alpha$. Equivalently, $d(\alpha)$ counts the number of elements of $X$ that are not attained by $\alpha$.

*Example.* If $X = \{1, 2, 3, 4\}$ and $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 4 \end{pmatrix}$, then $\operatorname{Im}(\alpha) = \{2, 4\}$, so $d(\alpha) = 4 - 2 = 2$.

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**Definition 2.9.** *[14][Gravity]*Let $\alpha \in T_n \setminus S_n$ (or $P_n \setminus S_n$). The *gravity* of $\alpha$, denoted $g(\alpha)$, is defined by

$$g(\alpha) = n + c(\alpha) - f(\alpha),$$

where $c(\alpha)$ is the number of cyclic orbits of $\alpha$ and $f(\alpha)$ is the number of acyclic or trivial orbits of $\alpha$.

*Example.* If $\alpha$ consists of one cycle $(1\,2\,3)$ and a fixed point 4, then $c(\alpha) = 1$, $f(\alpha) = 1$, so $g(\alpha) = 4 + 1 - 1 = 4$.

**Definition 2.10** (Local Depth)**.** Let $\alpha \in T_n$ be a quasi-idempotent. The *local depth* of $\alpha$ at $x \in X$, denoted $\mathrm{ldepth}_\alpha(x)$, is the least nonnegative integer $m$ such that $x\alpha^m \in K(\Omega)$, where $\Omega$ is the orbit of $x$ under $\alpha$ and $K(\Omega)$ is its eventual image. The *local depth* of $\alpha$ is the maximum of $\mathrm{ldepth}_\alpha(x)$ over all $x \in X$:

$$\mathrm{ldepth}(\alpha) = \max_{x \in X}\{\mathrm{ldepth}_\alpha(x)\}.$$

*Example.* If $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \end{pmatrix}$ on $X = \{1, 2, 3\}$, then $\mathrm{orbit}(1) = \{1, 2, 3\}$ and $K(\{1, 2, 3\}) = \{3\}$. Here, $1\alpha^2 = 3$, so $\mathrm{ldepth}_\alpha(1) = 2$, and similarly $\mathrm{ldepth}_\alpha(2) = 1$, $\mathrm{ldepth}_\alpha(3) = 0$. Hence $\mathrm{ldepth}(\alpha) = 2$.

**Definition 2.11.** [16] (Measure).
Let $\xi = [x_1, x_2, x_3]$ be an arbitrary 3-chain of $\alpha$. $\xi$ maps $x_1$ to $x_2$, $x_2$ to $x_3$, $x_3$ to $x_0$, and all other elements of $X_n^0$ identically. Then $\xi$ has a linear notation as

$$\xi = [x_1, x_2, x_3] = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \begin{pmatrix} x_3 \\ \emptyset \end{pmatrix}$$

because of the Vagner Representation Theorem. We shall refer to $x_1$ as the first entry, $x_2$ as the middle entry (or second entry), $x_3$ as the third entry, and $x_0$ as no entry of $\xi$. Let $\alpha \in P_n$. The equivalence relation $\sim = \{(x, y) \in X_n^0 \times X_n^0 : (\exists u, v \geq 0)\ x\alpha^u = y\alpha^v\}$ partitions $X_n^0$ into orbits $\Omega_1, \Omega_2, \ldots, \Omega_k$. These orbits correspond to the connected components of the digraph associated with $\alpha$ with vertex set $X_n^0$, in which there is a directed edge $(x, y)$ if and only if $x\alpha = y$. Each orbit $\Omega$ has a kernel defined by

$$K(\Omega) = \{x \in \Omega : (\exists r > 0)\ x\alpha^r = x\}.$$

Every orbit of $\alpha$ falls into exactly one of these five categories, and all five cases can arise for a single $\xi \in P_n$. Let $c(\alpha)$ be the number of cyclic orbits of $\alpha$ and $f(\alpha)$ be the number of fixed points of $\alpha$, which is equal to the sum of the number of terminal, trivial, and acyclic orbits of $\alpha$. The gravity of $\alpha$ is defined as

$$g(\alpha) = n + c(\alpha) - f(\alpha).$$

For each standard or acyclic orbit $\Omega$ of $\alpha \in P_n$ and each $x \in \Omega \backslash \mathrm{im}(\alpha)$, the sequence $x, x\alpha, x\alpha^2, x\alpha^3, \ldots$ eventually arrives in $K(\Omega)$ and remains there for all subsequent iterations. Denote the set of all distinct elements in this sequence by $Z(x)$. Suppose that $\alpha \in P_n \setminus S_n$ has $s$ standard orbits $\Omega_1, \Omega_2, \ldots, \Omega_s$. For each $j = 1, 2, \ldots, s$, let $\Omega_j \setminus \mathrm{im}(\alpha) = \{x_{1j}, x_{2j}, \ldots, x_{k_j j}\}$, where $x_{1j}$ is such that

$$|Z(x_{1j})| = \begin{pmatrix} \max_{1 \leq i \leq k_j}\{|Z(x_{ij})| : |Z(x_{ij})| \text{ is even}\} \\ \max_{1 \leq i \leq k_j}\{|Z(x_{ij})| : |Z(x_{ij})| \text{ is odd}\} \end{pmatrix}.$$

Then there exist $m_j \geq 1$ and $r_j \geq 2$ such that

$$K(\Omega_j) = \{x_{1j}\alpha^{m_j}, x_{1j}\alpha^{m_j+1}, \ldots, x_{1j}\alpha^{m_j+r_j-1}\}, \quad \text{where} \quad x_{1j}\alpha^{m_j+r_j} = x_{1j}\alpha^{m_j}.$$

Note that this definition of $K(\Omega_j)$ is still valid for every $x_{ij}$, not only for $x_{1j}$, and moreover, they are all the same. Let

$$Z_1(\Omega_j) = Z(x_{1j}) = \{x_{1j}, x_{1j}\alpha, \ldots, x_{1j}\alpha^{m_j}, x_{1j}\alpha^{m_j+1}, \ldots, x_{1j}\alpha^{m_j+r_j-1}\}$$

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and

$$Z_i(\Omega_j) = \{x_{ij}, x_{ij}\alpha, \ldots, x_{ij}\alpha^{p_{ij}-1}\}, \quad (2 \le i \le k_j),$$
$$\text{where} \quad x_{ij}\alpha^{p_{ij}} \in \big(Z_1(\Omega_j) \cup Z_2(\Omega_j) \cup Z_3(\Omega_j) \cup \ldots \cup Z_{i-1}(\Omega_j)\big).$$

Then $\{Z_i(\Omega_j) : 1 \le i \le k_j\}$ is a partition of $\Omega_j$. Suppose that $\alpha \in P_n \setminus S_n$ has an acyclic or terminal orbit; let $\Phi$ be the union of all its acyclic orbits and let $\Xi$ be the union of all its terminal orbits, and denote the set $\{x \in \Phi : x\alpha = x\}$ by $\text{Fix}(\Phi)$. Let $\Phi \setminus \text{im}(\alpha) = \{x_1, x_2, \ldots, x_l\}$ where; $x_1$ is such that

$$|Z(x_1)| = \begin{pmatrix} \max_{1\le i\le k_j}\{|Z(x_u)| : |Z(x_u)| \text{ is even}\} \\ \max_{1\le i\le k_j}\{|Z(x_u)| : |Z(x_{ij})| \text{ is odd}\} \end{pmatrix}.$$

Then, for $u = 1, 2, \ldots, l$ define

$$Y_u(\Phi) = \{x_u, x_u\alpha, \ldots, x_u\alpha^{q_u-1}\},$$

where

$$x_1\alpha^{q_1} \in \text{Fix}(\Phi)$$

and

$$x_u\alpha^{q_u} \in (Y_1(\Phi) \cup Y_2(\Phi) \cup \ldots \cup Y_{u-1}(\Phi) \cup \text{Fix}(\Phi)) \quad (u = 2, 3, \ldots, l).$$

Thus, $\{Y_u(\Phi) : 1 \le u \le l\}$ is a partition of $\Phi$. We will be interested in the cardinalities of $Z_i(\Omega_j)$ and $Y_u(\Phi)$ being odd or even. For this, we define indicator functions $z_{ij}$ and $y_u$ by

$$z_{ij} = \begin{pmatrix} 0 & \text{if } |Z_i(\Omega_j)| \text{ is even} \\ 1 & \text{if } |Z_i(\Omega_j)| \text{ is odd} \end{pmatrix},$$

$$y_u = \begin{pmatrix} 0 & \text{if } |Y_u(\Phi)| \text{ is even} \\ 1 & \text{if } |Y_u(\Phi)| \text{ is odd} \end{pmatrix},$$

and

$$w_u = \begin{pmatrix} 0 & \text{if } |W_u(\Xi)| \text{ is even} \\ 1 & \text{if } |W_u(\Xi)| \text{ is odd} \end{pmatrix}.$$

For each $\alpha \in P_n \setminus S_n$, we define the measure of $\alpha$ by

$$m(\alpha) = \begin{pmatrix} l(\alpha) - e(\alpha) & \text{if } l(\alpha) > e(\alpha) \\ 0 & \text{if } l(\alpha) \le e(\alpha) \end{pmatrix},$$

where

$$l(\alpha) = \sum_{j=1}^{s}\sum_{i=1}^{k_j} z_{ij} + \sum_{u=1}^{l} y_u + \sum_{v=1}^{b} w_u$$

and $e(\alpha)$ denotes the number of cyclic orbits of $\alpha$ of even cardinality.

**Definition 2.12.** Let $\alpha \in T_n$. In the standard way, we define

$$\text{im}\,\alpha = \{x\alpha : x \in X\}, \quad \text{rank}\,\alpha = |\text{im}\,\alpha|, \quad \ker\alpha = \{(x, y) \in X \times X : x\alpha = y\alpha\},$$
$$\text{fix}\,\alpha = \{x \in X : x\alpha = x\}.$$

As in [5], the key idea is that of an *orbit* of $\alpha$, i.e. an equivalence class in $X$ under the equivalence

$$\omega = \{(x, y) \in X \times X : (\exists l, m \ge 0)(x\alpha^l = y\alpha^m)\}.$$

It is shown in [5] that each orbit $\Omega$ has a *kernel* $K(\Omega)$ characterised by the property that (for each $x \in \Omega$)

$$x \in K(\Omega) \quad \text{if and only if} \quad x \in x\alpha^{-N},$$

where

$$x\alpha^{-N} = \{y \in X : y\alpha^i = x \text{ for some } i > 0\}.$$

Orbits are then classified into various types:

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- **standard orbits**: $|\Omega| > |K(\Omega)| > 1$;
- **acyclic orbits**: $|\Omega| > |K(\Omega)| = 1$;
- **cyclic orbits**: $|\Omega| = |K(\Omega)| > 1$;
- **singleton orbits**: $|\Omega| = |K(\Omega)| = 1$.

**Note:** Orbits are the connected components of the associated digraph in a full transformation semigroup. For instance, let $\alpha \in T_{14}$ be given as;

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 3 & 3 & 4 & 5 & 6 & 4 & 6 & 9 & 10 & 10 & 12 & 13 & 11 & 14 \end{pmatrix}.$$

The associated digraph is;

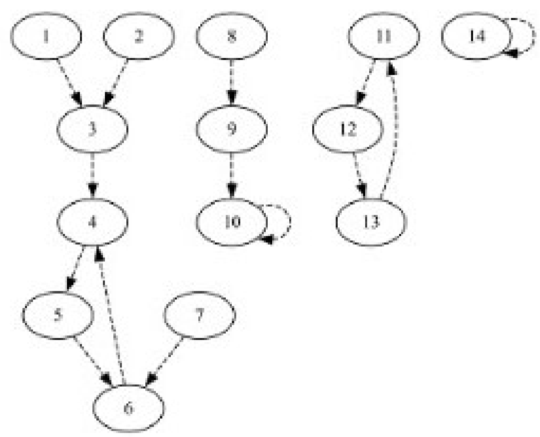


Figure 1: Connected Orbits in $T_n$

The example illustrates the various types of orbits that can arise. The leftmost orbit is *standard*, followed by an *acyclic* orbit, then a *cyclic* orbit, and finally a *trivial* orbit.

## 3 Main Results

**Definition 3.1.** Let $X$ be a finite set of size $n$. The *full transformation semigroup* on $X$ is denoted by $T_n$, and the *partial transformation semigroup* on $X$ is denoted by $P_n$.

**Definition 3.2.** An element $\alpha \in T_n$ (or $P_n$) is called an *idempotent* if $\alpha^2 = \alpha$. We say that $\alpha$ is a *quasi-idempotent* if there exist integers $k > \ell \geq 1$ such that $\alpha^k = \alpha^\ell$. If, moreover, there exists a minimal $m \geq 1$ with $\alpha^{m+1} = \alpha^m$, then $\alpha$ is called a *stable quasi-idempotent* (it stabilizes after $m$ iterations).

**Definition 3.3.** For $\alpha \in T_n$, define a relation $\sim$ on $X$ by

$$x \sim y \iff \exists\, r, s \geq 0 \text{ such that } x\alpha^r = y\alpha^s.$$

Let $\{\Omega_i\}$ be the equivalence classes (orbits) of $\sim$. For each orbit $\Omega_i$, we denote by $K(\Omega_i)$ the set of points in $\Omega_i$ that lie in the eventual image, i.e., the points that are fixed by some power of $\alpha$.

**Theorem 3.4.** The stable quasi-idempotent of defect $d$ in $P_n$ or $T_n$ has diameter or longest trajectory as 2.

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*Proof.* Let $\Gamma = (V, E)$ be a digraph. Then $E \subseteq X_n \times X_n$ and it is a set of ordered pairs $(a, b)$. The set $E$ is the set of edges, while $V$ is the set of vertices. Axiomatically, the graph $\Gamma$ is a set $V$ equipped or enclosed or endowed with a relation $E$ which is reflexive and symmetric. Since $(a, b) \in E$ and

$$\alpha = \begin{pmatrix} (a,b) & (a,b+1) & (a,b+2) \\ (a,b+1) & (a,b+2) & (a,b+2) \end{pmatrix} \in T_n \text{ or } P_n.$$

Then the longest trajectory is $(a, b+2) - (a, b) = 2$. Let $b^* = (a, b), b^* + 1 = (a, b+1), b^* + 2 = (a, b+2)$. Then $\alpha^* = [b^*, b^* + 1, b^* + 2][b^* + 3]$ is a 3-path which is a stable quasi-idempotent. Since $\beta^* = [b^*, b^* + 1, b^* + 2, b^* + 3][b^* + 3]$ gives $\beta^* + \beta^2 \neq \beta^3 = \beta^4$. Then the longest diameter or trajectory is 2. □

**Theorem 3.5** (Decomposition of Stable Quasi-Idempotents)**.** Let $\alpha \in T_n$ (or $P_n$) be a stable quasi-idempotent of defect $d+\delta$. Then $\alpha$ can be decomposed as a product of two stable quasi-idempotents: one of defect $d$ and one of defect $\delta$.

*Proof.* Let $\alpha$ be represented as $\gamma_{x,y}(w)$ for some parameters $x, y, w$ to be specified. Then we may write

$$\gamma_{x,y}(w) = \gamma_{x,y}(z) \circ \gamma_{x,y}(u),$$

where $\gamma_{x,y}(z)$ is a stable quasi-idempotent of defect $d$ and $\gamma_{x,y}(u)$ is a stable quasi-idempotent of defect $\delta$. Since the composition of stable quasi-idempotents preserves stability, the resulting map is again a stable quasi-idempotent. Therefore $\alpha$ decomposes as required. □

**Theorem 3.6.** The rank of stable quasi-idempotent having $|\text{span}(\alpha)| = s$ in $T_n$ is $\frac{n!}{(n-s)!s!}$.

*Proof.* Let $|\text{span}(\alpha)| = |\text{dom}(\alpha) \cup \text{im}(\alpha)|$. Since an idempotent is $[x_0, x_1|x_1]$ if the origin of the digraph $\Gamma = (V, E)$ is involved, or $\epsilon = [x_1, x_2|x_2]$ using the definition of the $(m, r)$-path cycle. Then $\text{span}(\epsilon) = \{x_1, x_2\}$, that is, $|\text{span}(\epsilon)| = 2$, which gives $\frac{n!}{(n-2)!2!} = \frac{n(n-1)}{2} = S(n, n-1)$, a Stirling number of the second kind [17].
Since the order of the span of

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_3 \end{pmatrix}$$

is given by $|\text{span}(\xi)| = |\{x_1, x_2, x_3\} \cup \{x_2, x_3\}| = 3$. Then, the rank of the stable quasi-idempotent of $|\text{span}(\xi)| = s$ is $\frac{n!}{(n-s)!s!}$. □

**Theorem 3.7.** The local depth of stable quasi-idempotent of defect 1 in $T_n$ is less than or equal to $\frac{g(\alpha)+3}{2}$.

*Proof.* For each $\alpha$ in $S$ the least $k$ for which $\alpha \in E^k$ is $k = \frac{g(\alpha)+3}{2}$, where $g(\alpha)$ is the gravity of $\alpha$. We shall write the element $\xi$ of $E_2$ given, for each pair of distinct $i$ and $j$ in $X$, by $i\xi = j$, $j\xi = l$, $x\xi = \xi(x \neq i)$ as $\xi = \begin{pmatrix} i & j \\ j & l \end{pmatrix}$ for the stable quasi-idempotent $\xi$. We shall refer to $i$ and $j$ as middle entries and $l$ as the lower entry of $\xi$. Let us suppose that $\alpha$ has orbits as follows:

*Standard orbits:* $\Omega_1, \Omega_2, \ldots, \Omega_{m/2}$;

*Acyclic orbits:* $\Omega_{m/2+1}, \Omega_{m/2+2}, \ldots, \Omega_{m/2+a/2}$;

*Cyclic orbits:* $\Omega_{m/2+a/2+1}, \Omega_{m/2+a/2+2}, \ldots, \Omega_{m/2+a/2+c/2}$;

*Trivial orbits:* $\Omega_{m/2+a/2+c/2+1}, \Omega_{m/2+a/2+c/2+2}, \ldots, \Omega_{m/2+a/2+c/2+s/2}$.

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Then $\alpha$ has $\frac{a+s}{2}$ fixed points. Also, $\sum_{i=1}^{m/2+a/2+c/2+s/2}|\Omega_i| = n$ and so $\sum_{i=1}^{m/2+a/2+c/2}|\Omega_i| = \frac{n}{2} - \frac{s}{2}$.
We shall show that

$$\alpha = \beta_1 \dots \beta_{m/2}\gamma_{m/2+1}\cdots\gamma_{m/2+a/2}\delta_{m/2+a/2+1}\dots\delta_{m/2+a/2+c/2},$$

where each $\beta_i$ is a product of $\frac{|\Omega_i|}{2}$ stable quasi-idempotent, each $\gamma_i$ is a product of $\frac{|\Omega_i|-1}{2}$ stable quasi-idempotent, and each $\delta_i$ is a product of $\frac{|\Omega_i|+1}{2}$ stable quasi-idempotent.

The total number of stable quasi-idempotent used is thus;

$$\sum_{i=1}^{m/2}|\Omega_i| + \sum_{i=1}^{m/2+a/2}(|\Omega_i|-1) + \sum_{i=1}^{m/2+a/2+c/2}(|\Omega_i|+1) = \frac{n}{2} + \frac{c}{2} - \frac{(a/2+s/2)}{2}$$

(using $\sum_{i=1}^{m/2+a/2+c/2}|\Omega_i| = \frac{n}{2} - \frac{s}{2}$),

which equals $\frac{g(\alpha)}{2}$.

To avoid excessive use of subscripts, let us now consider a typical orbit $\Omega$ from $\{\Omega_1,\dots,\Omega_m\}$. Let $x_1 \in \Omega\setminus \text{ran}\alpha$ and let $K(\Omega) = \{x_1\alpha^m, x_2\alpha^{m+1},\dots,x_1\alpha^{m+r-1}\}$, where $x_1\alpha^{(m/2+r)} = x_1\alpha^{(m/2)}$.
Let

$$\zeta_1 = \begin{pmatrix} x_1\alpha^{(m+r-1)} & x_1\alpha^{(m+r-2)} \\ x_1\alpha^{(m+r-2)} & x_1\alpha^{(m-1)} \end{pmatrix}\begin{pmatrix} x_1\alpha^{(m+r-3)} & x_1\alpha^{(m+r-2)} \\ x_1\alpha^{(m+r-2)} & x_1\alpha^{(m+r-3)} \end{pmatrix}\cdots\begin{pmatrix} x_1\alpha^2 & x_1\alpha^3 \\ x_1\alpha^3 & x_1\alpha \end{pmatrix}$$

a product of $\frac{(m+r)}{2}$ stable quasi-idempotent, and let $Z_1$ be the set of elements of $\Omega$ appearing along the top of this product: $Z_1 = \{x_1, x_1\alpha,\dots,x_1\alpha^{(m+r-1)}\}$.

If $Z_1 = \Omega$, then $\zeta_1$ is the product $\beta$ of idempotents we require; otherwise choose $x_2(\neq x_1)$ in $\Omega\setminus\text{ran}\alpha$. Since $K(\Omega)\subseteq Z_1$, there is a least $p$ such that $x_2\alpha^p \in Z_1$. Let

$$\zeta_2 = \begin{pmatrix} x_1\alpha^{(p-1)} & x_1\alpha^{(p-2)} \\ x_1\alpha^{(p-2)} & x_1\alpha^{p} \end{pmatrix}\begin{pmatrix} x_1\alpha^{(p-3)} & x_1\alpha^{(p-4)} \\ x_1\alpha^{(p-4)} & x_1\alpha^{(p-2)} \end{pmatrix}\cdots\begin{pmatrix} x_2\alpha^2 & x_2\alpha^3 \\ x_2\alpha^3 & x_2\alpha \end{pmatrix}$$

and let $Z_2 = \{x_2, x_2\alpha,\dots,x_2\alpha^{(p-1)}\}$. If $Z_1\cup Z_2 = \Omega$, then $\beta = \zeta_1\zeta_2$ is the product of stable quasi-idempotent we require; otherwise we continue, finding

$$\zeta_3 = \begin{pmatrix} x_3\alpha^{(q-1)} & x_3\alpha^{(q-2)} \\ x_3\alpha^{(q-2)} & x_3\alpha^{q} \end{pmatrix}\begin{pmatrix} x_3\alpha^{(q-3)} & x_3\alpha^{(q-2)} \\ x_3\alpha^{(q-2)} & x_3\alpha^{(q-4)} \end{pmatrix}\cdots\begin{pmatrix} x_3\alpha^2 & x_3\alpha^3 \\ x_3\alpha^3 & x_3\alpha \end{pmatrix}$$

and $Z_3 = \{x_3, x_3\alpha,\dots,x_3\alpha^{(m+r-1)}\}$; and so on. Eventually we have $\Omega$ as the disjoint union of $Z_1,\dots,Z_k$ and a product $\beta = \zeta_1\dots\zeta_k$ of $|Z_1|+\cdots+|Z_k| = |\Omega|$ stable quasi-idempotent.

Now notice that each element $z$ of $\Omega$ appears exactly once as an upper entry in the product $\zeta_1\zeta_2\zeta_3\dots\zeta_k$. Moreover, with the sole exception of $z = x_1\alpha^{(m-1)}$, an element $z$ appearing as a lower entry never subsequently reappears as an upper entry. Hence each $z \neq x_1\alpha^{(m+r-1)}$ is moved by exactly one of the stable quasi-idempotent appearing in $\beta = \zeta_1\zeta_2\zeta_3\dots\zeta_k$, and it is moreover moved to $z\alpha$. The exceptional element $x_1\alpha^{(m+r-1)}$ is moved to $x_1\alpha^{(m-1)}$ by the first stable quasi-idempotent in the product $\zeta_1$, and then is moved by $\begin{pmatrix} x_1\alpha^{(m+r-1)} & x_1\alpha^{(m+r-2)} \\ x_1\alpha^{(m+r-2)} & x_1\alpha^{(m-1)} \end{pmatrix}$ to $x_1\alpha^m(=x_1\alpha^{(m+r)})$. Thus $z\beta = z\alpha$ for every $z$ in $\Omega$, while $x\beta = x$ for every $x\notin\Omega$.

This argument applies to each of the standard orbits $\Omega_1,\dots,\Omega_m$. If we now suppose that $\Omega$ is a cyclic orbit, then exactly the same argument applies, except that the formula

$$\zeta_1 = \begin{bmatrix} x_1\alpha^{(m+r-1)} & x_1\alpha^{(m-1)} \\ x_1\alpha^{(m-1)} & x_1 \end{bmatrix}\begin{bmatrix} x_1\alpha^{(m+r-3)} & x_1\alpha^{(m+r-2)} \\ x_1\alpha^{(m+r-2)} & x_1\alpha^{(m+r-4)} \end{bmatrix}\cdots\begin{bmatrix} x_1\alpha^2 & x_1\alpha^3 \\ x_1\alpha^3 & x_1 \end{bmatrix}$$

simplifies to

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$$\zeta_1 = \begin{bmatrix} x_1\alpha^{(m-1)} & x_1\alpha^{(m-2)} \\ x_1\alpha^{(m-2)} & x_1 \end{bmatrix} \begin{bmatrix} x_1\alpha^{(m-3)} & x_1\alpha^{(m-2)} \\ x_1\alpha^{(m-2)} & x_1\alpha^{(m-4)} \end{bmatrix} \cdots \begin{bmatrix} x_1\alpha^2 & x_1\alpha^3 \\ x_1\alpha^3 & x_1 \end{bmatrix},$$

a product of $\frac{m}{2}$ stable quasi-idempotents. (In effect, it is not necessary to provide a stable quasi-idempotent to specify the image of $x_1\alpha^m$, since this is a fixed point.) We thus obtain a product $\gamma$ of $\frac{|\Omega|-1}{2}$ stable quasi-idempotents with the property that $z\gamma = z\alpha$ for all $z \in \Omega$, while $x\gamma = x$ for all $x \notin \Omega$.

Now suppose that we have found $\beta_1 \ldots \beta_{(m/2)}\gamma_{(m/2+1)} \cdots \gamma_{(m+a/2)}$. Then the last stable quasi-idempotent in the product $\beta_1 \ldots \beta_{(m/2)}\gamma_{(m/2+1)} \cdots \gamma_{(m+a/2)}$ is

$$\begin{pmatrix} y\alpha & y\alpha^2 \\ y\alpha^2 & y \end{pmatrix},$$

where $y \notin \text{ran}\alpha$. Let $\Omega = \{x, x\alpha, \ldots, x\alpha^{(r-1)}\}$ be a cyclic orbit, where $x\alpha^r = x$. Define

$$\delta = \begin{pmatrix} x\alpha^{(r-1)} & y \\ y & x\alpha^{(r-2)} \end{pmatrix} \cdots \begin{pmatrix} x & x\alpha \\ x\alpha & y \end{pmatrix}$$

as a product of $\frac{(|\Omega|+1)}{2}$ stable quasi-idempotent. Now $y \notin \text{ran}(\beta_1 \ldots \beta_{(m/2)}\gamma_{(m/2+1)} \cdots \gamma_{(m/2+a/2)})$ and so $z\delta = z$ for all $z$ in $\text{ran}(\beta_1 \ldots \beta_{(m/2)}\gamma_{(m/2+1)} \cdots \gamma_{(m/2+a/2)}) \setminus \Omega$. Also $z\delta = z\alpha$ for all $z$ in $\Omega$. Doing this (using the same $y$) for each of the cyclic orbits yields

$$z\alpha = z\beta_1 \ldots \beta_{(m/2)}\gamma_{(m/2+1)} \cdots \gamma_{(m/2+a/2)}\delta_{(m/2+a/2+1)} \cdots \delta_{(m/2+a/2+c)}$$

for all $z$ in $X\setminus$ since clearly the fixed points of $\alpha$ are left fixed by both sides. Thus $\alpha$ is expressible as a product of $\frac{g(\alpha)+3}{2}$ stable quasi-idempotent at most. □

**Theorem 3.8.** Let span($\alpha$) be the number of entries of $\alpha \in T_n \setminus S_n$. Let

$$\alpha = \prod_{j=1}^{n/m} [a_{1j}, a_{2j}, a_{3j}, \ldots, a_{(m-1)j} \mid a_{mj}]$$

whenever span($\alpha$) is even, and

$$\alpha = \prod_{j=1}^{n/m} [a_{1j}, a_{2j}, a_{3j}, \ldots, [a_{(m-1)j}, a_{(m-2)j}] \mid a_{mj}]$$

whenever span($\alpha$) is odd. That is,

$$\alpha = \epsilon_1\epsilon_2\epsilon_3 \ldots \epsilon_{k(\alpha)},$$

where $|\text{span}(\epsilon_i)| = m$ (where $\epsilon_i$ is a SQI). Then,

$$\max k(\alpha) = \left\lceil \frac{n - D(\alpha)}{\text{span}(\epsilon_i)} \right\rceil.$$

*Proof.* Since $T_n \subseteq B_n$, i.e., the transformation semigroup is a subsemigroup of the binary relation semigroup, then span($\alpha$) is even implies span($\alpha$) + 1 is odd or span($\alpha$) − 1 is even, which implies $e_n(\alpha)$ is odd. Since

$$\alpha = \prod_{j=1}^{n/m} [a_{1j}, a_{2j}, a_{3j}, \ldots, [a_{(m-2)j}, a_{(m-1)j}] \mid a_{mj}]$$

$$= \prod_{j=1}^{n/m} [[a_{1j}, a_{1j}], [a_{2j}, a_{2j}], \ldots, [a_{(m-2)j}, a_{(m-1)j}] \mid [a_{mj}, a_{mj}]].$$

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such that not all the entries appear in the linear notation, then $\operatorname{span}(\alpha)$ exists irrespective. This is also because $\alpha$ is a product of an even number of $m$-paths (transpositions for the trivial case $\operatorname{span}(\epsilon_i) = 1$) and the elements of $X_n$ are placeable such that $\operatorname{dom}(\alpha) \cap \operatorname{im}(\alpha) = \emptyset$ as

$$\begin{pmatrix} \operatorname{dom}(\alpha) \\ \operatorname{im}(\alpha) - D(\alpha) \end{pmatrix} = (\operatorname{dom}(\alpha), \operatorname{im}(\alpha) - D(\alpha)) \subseteq B_n$$

if $|\{a_{11}, \ldots, a_n\}|$ is odd, i.e., $n = 2m+1$, then $|\{a_{11}, \ldots, \{a_{n-1}, a_n\}\}|$ is even, i.e., $n = 2m+1-1 = 2m$ and $\alpha_2$ is well-defined.

If $|\{a_{11}, \ldots, a_n\}|$ is even, then it is well-defined.

We can now prove this result inductively. If $\operatorname{span}(\epsilon_i) = 1$, the result follows trivially as a product of identity idempotents, viz: $(1)(2)(3)(4)\ldots$. Assume that $\operatorname{span}(\epsilon_i) = \kappa - 1$ and let

$$\alpha = \prod_{j=1}^{k-1} \left[a_{1j}, a_{2j}, a_{3j}, \ldots, \left[a_{(m-1)j}, a_{(m-2)j}\right] \mid a_{mj}\right].$$

Then;

$$\alpha = \left[a_{11}, a_{21}, a_{31}, \ldots, \left[a_{(m-1)1}, a_{(m-2)1}\right] \mid a_{m1}\right] \ldots \left[a_{1\kappa}, a_{2\kappa}, a_{3\kappa}, \ldots, \left[a_{(m-1)\kappa}, a_{(m-2)\kappa}\right] \mid a_{m\kappa}\right]$$

$$\Rightarrow \alpha = \left[a_{11}, a_{21}, a_{31}, \ldots, a_{(m-1)1}, a_{(m-2)1} \mid a_{m1}\right] \ldots \left[a_{1\kappa}, a_{2\kappa}, a_{3\kappa}, \ldots, a_{(m-1)\kappa}, a_{(m-2)\kappa} \mid a_{m\kappa}\right]$$

$$\Rightarrow \alpha = \prod_{j=1}^{k} \left[a_{1j}, a_{2j}, a_{3j}, \ldots, \left[a_{(m-1)j}, a_{(m-2)j}\right] \mid a_{mj}\right],$$

i.e., $\operatorname{span}(\epsilon_i) = \kappa$. Since by the well-ordering property of natural numbers $n$, for $l \leq n - D(\alpha)$, then

$$\left\lceil \frac{n - D(\alpha)}{\kappa} \right\rceil$$

is a natural number. Thus, there exist products of idempotents of arbitrary entries $\operatorname{span}(\epsilon_i) = k$ for every $n$ $(\forall \alpha \in T_n \setminus S_n)$. When $D(\alpha) = 0$, we retrieve the global depth as

$$\max k(\alpha) = \left\lceil \frac{n - D(\alpha)}{\kappa} \right\rceil.$$

□

**Corollary 3.9.** Let $\operatorname{span}(\alpha)$ be the $\operatorname{dom}(\alpha) \cup \operatorname{im}(\alpha)$ for $\alpha \in P_n \setminus S_n$; let

$$\alpha = \prod_{j=1}^{n/m} \left[a_{1j}, a_{2j}, a_{3j}, \ldots, a_{(m-1)j} \mid a_{mj}\right]$$

when $\operatorname{span}(\alpha)$ is even, and

$$\alpha = \prod_{j=1}^{n/m} \left[a_{1j}, a_{2j}, a_{3j}, \ldots, \left[a_{(m-1)j}, a_{(m-2)j}\right] \mid a_{mj}\right]$$

when $\operatorname{span}(\alpha)$ is odd. If

$$\alpha = \epsilon_1 \epsilon_2 \epsilon_3 \ldots \epsilon_{k(\alpha)},$$

then

$$\max k(\alpha) \leq \left\lceil \frac{n+1}{\operatorname{span}(\epsilon_i)} \right\rceil,$$

where $\epsilon_i$ is an idempotent and $|\operatorname{span}(\epsilon_i)| = m$.

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**Corollary 3.10.** Let span($\alpha$) be the number of entries of $\alpha \in P_n \setminus S_n$; let

$$\alpha = \prod_{j=1}^{n/m} [a_{1j}, a_{2j}, a_{3j}, \ldots, a_{(m-1)j} \mid a_{mj}]$$

whenever span($\alpha$) is even, and

$$\alpha = \prod_{j=1}^{n/m} [a_{1j}, a_{2j}, a_{3j}, \ldots, [a_{(m-1)j}, a_{(m-2)j}] \mid a_{mj}]$$

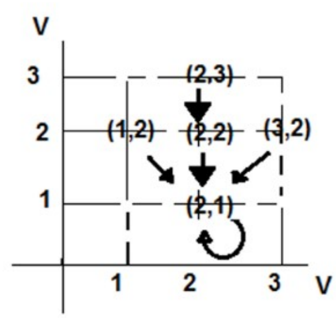
whenever span($\alpha$) is odd. That is,

$$\alpha = \epsilon_1 \epsilon_2 \epsilon_3 \ldots \epsilon_{k(\alpha)},$$

where $|\text{span}(\epsilon_i)| = m$ (where $\epsilon_i$ is a SQI). Then

$$\max k(\alpha) \leq \left\lceil \frac{(n+1) - d(\alpha)}{\text{span}(\epsilon_i)} \right\rceil.$$

**EXAMPLE 1:** Digraph of $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \end{pmatrix}$



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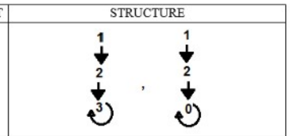
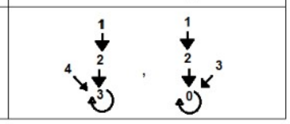
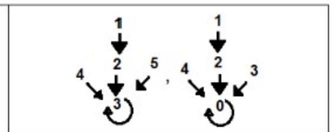
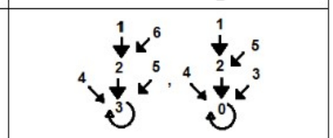
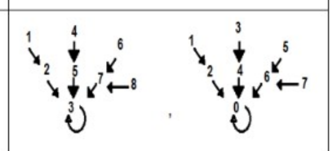
**Example 2:** DEFECTS OF STABLE QUASI-IDEMPOTENTS

| DEFECT | STRUCTURE |
|---|---|
| 1 |  |
| 2 |  |
| 3 |  |
| 4 |  |
| 4 |  |
| ⋮ | ⋮ |

# 4 Competing Interests

The authors declare that they have no competing interests.

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