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A Generalized Differential Operator in Function Theory with Applications to Coefficient Bounds of p-Valent Analytic Functions

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Abstract

In this paper, a new differential operator that generalizes several well-established operators in geometric function theory by incorporating principles of q-calculus and p-valent analytic functions is introduced. The key objectives include establishing its equivalence to existing operators and deriving coefficient bounds for the associated p-valent function classes. Using q-differentiation and multiplier transformations, we formulate a generalized class of analytic functions and derive coefficient bounds within the unit disk. Numerical comparisons and graphical illustrations reveal that the new operator yields finer results.

Keywords: p-Valent, q-Number, q-Derivatives, Analytic functions, Integral operator.
MSC2010: 30C45.

1 INTRODUCTION

Let **A** denote the class of normalized univalent functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ with $f(0) = 0$ and $f'(0) = 1$. Recent advances in q-calculus have significantly influenced the theory of geometric functions, particularly in the study of differential operators and analytic functions. Originating from quantum calculus, q-derivatives and q-numbers provide a discrete framework that converges to classical calculus as $q \to 1$. This duality has allowed researchers to explore generalizations of analytic functions classes, including star-like, convex, and p-valent functions. Central to this work are q-numbers, defined for $q \in (0,1)$, which interpolate between discrete and continuous systems. The author of [1] introduces the q-difference operator and discussed some applications of the q-derivative and the q-integral. Several authors [2–5] applied properties of the q-difference operator to discuss subclasses of complex analytic functions. Studies by authors of [6,7] established foundational principles q-calculus, while authors of [8–10] developed multiplier differential operators for univalent analytic functions.

For functions $f(z)$ as defined in (1.1), the coefficients a_n play a prominent role in the study of analytic functions. Thus, in applying analytic function to any area of life, it is important to study their coefficient bounds. Recent studies have focused on obtaining sharp coefficient bounds for specialized subclasses of analytic functions. For example, the authors of [11] investigated Bazilevic-type functions associated with conic domains, while the authors of [12] introduced new subclasses defined with respect to symmetric and conjugate points via an extended Salagean derivative operator. The author in [13] stated that starlike and convex functions which are aspects of analytic functions have their applications in human physiology, physical and natural phenomena. The author in [14] applies analytic functions to prevent a population of disease carriers from being transformed into an infectious population, thus reducing the risk of spread of the disease. The author in [14] used the operator defined in (2.6) to obtain (2.7).

This paper extends these efforts by proposing a unified operator $I^m_{q,\vartheta,\alpha,\beta,\gamma}$ that incorporates q-differentiation, p-valency, and multiple parameters $(\alpha,\beta,\gamma,\vartheta)$. We demonstrate its capacity to generalize the Salagean, Al-Oboudi, and Makinde operators through parameter specialization. Furthermore, the operator also yields coefficient bounds of less magnitude than the one obtained by the author of [14]. We begin by presenting the concept of q-number and q-differentiation.

Definition 1.1. ***(q-number)****: Let $q \in (0,1)$. The q-number of n is defined as*

$$[n]_q = \begin{cases} 0, & \text{if } n = 0, \\ \sum\limits_{k=0}^{n-1} q^k, & \text{if } n \in \mathbb{N}, \\ \frac{1-q^n}{1-q}, & \text{for } n \in \mathbb{R} \text{ or } \mathbb{C}, \text{ and } q \neq 1. \end{cases}$$

Furthermore, by L'Hopital's rule, we have:

$$\lim_{q\to 1}[n]_q = n, \quad \text{and} \quad \lim_{n\to\infty}[n]_q = \frac{1}{1-q}, \quad \text{for } q \in (0,1).$$

Note : Let $q \in (0,1)$ then the q-factorial is defined as follows;

$$[n]_q! = 0 \quad \text{for } n = 0,$$

and

$$[n]_q! = \prod_{k=1}^{n} [k]_q$$

if $n \in \mathbf{N}$. The q-difference (q-derivative or Jackson's derivative) operator D_q is defined as follows;

Definition 1.2. (**q-Derivative**): Let $f(z)$ be differentiable in the domain $\mathbb{D} \subset \mathbb{C}$ and let $q \in (0,1)$, then the q-derivative is defined by

$$D_q f(z) = f'(0)$$

for z = 0 and $f'(0)$ exist,

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}$$

for $z \neq 0$,

$$D_q^2 f(z) = D_q(D_q f(z))$$

and

$$D_q^m f(z) = D_q(D_q^{m-1} f(z))$$

(For more details check [1,6,7,16]).

Note that

$$\lim_{q\to 1} D_q f(z) = \lim_{q\to 1} \frac{f(z)-f(qz)}{(1-q)z} = f'(z)$$

where $f'(z)$ is the classical differentiation.

2 PRELIMINARIES

Let f be in **A**, then f is said to be starlike respectively convex if and only if

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \tag{2.1}$$

and

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right) > 0.(z\in U) \tag{2.2}$$

From (2.1) and (2.2) we have that $f(z)$ is convex if and only if $zf'(z)$ is starlike.
Swamy [10] introduced a multiplier differential operator of the form $I_{\alpha,\beta}$, where $\alpha,\beta \geq 0$ with $\alpha+\beta > 0$ defined by

$$I_{\alpha,\beta} f(z) = \frac{\alpha f(z) + \beta z f'(z)}{\alpha+\beta} \tag{2.3}$$

$$I_{\alpha,\beta}^m f(z) = I_{\alpha,\beta}(I_{\alpha,\beta}^{m-1} f(z)) \tag{2.4}$$

$$I_{\alpha,\beta}^m f(z) = z + \sum_{n=2}^{\infty}\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^m a_n z^n. \tag{2.5}$$

Makinde et al [9] introduced a multiplier differential operator of the form $I_{\alpha,\beta,\gamma}$, where $\alpha,\beta,\gamma \geq 0$ with $\alpha+\beta+\gamma > 0$ defined by

$$I_{\alpha,\beta,\gamma} f(z) = \frac{\alpha f(z) + \beta z f'(z) + \gamma z(zf'(z))'}{\alpha+\beta+\gamma} \tag{2.6}$$

$$I_{\alpha,\beta,\gamma}^m f(z) = I_{\alpha,\beta,\gamma}(I_{\alpha,\beta,\gamma}^{m-1} f(z))$$

$$I_{\alpha,\beta,\gamma}^m f(z) = z + \sum_{n=2}^{\infty}\left(\frac{\alpha+n\beta+n^2\gamma}{\alpha+\beta+\gamma}\right)^m a_n z^n. \tag{2.7}$$

In 2019, the author in [8] defined an integral operator and a class of p-valent analytic functions as follow:

$$F_{p,\mu}(z) = \int_0^z \prod_{i=1}^{k}\left(\frac{I_{\alpha,\beta,\gamma,}^m f_{i,p}(t)}{t}\right)^{\frac{1}{\mu}}$$

where $|\mu| < 1, \mu \in \mathbb{C}, 1 \leq i \leq k, \alpha,\beta,\gamma \geq 0$ with $\alpha+\beta+\gamma > 0, p \in \mathbb{N}$

$$I_{\alpha,\beta,\gamma,}^m f_{i,p}(z) = z^p + \sum_{n=p+1}^{\infty}\left(\frac{\alpha+n\beta+n^2\gamma}{\alpha+\beta+\gamma}\right)^m a_n^i z^n.$$

and the class $\Gamma_{p,\mu,\alpha,\beta\gamma}(\xi_1,\xi_2,\delta)$ defined as

$$\Gamma_{p,\mu,\alpha,\beta\gamma}(\xi_1,\xi_2,\delta) = \left\{ I_{\alpha,\beta,\gamma,}^m f_{i,p}(t) \in A : \left|\frac{H(z)+\frac{1}{\mu}-1}{\xi_1(H(z)+\frac{1}{\mu})+\xi_2}\right| < \delta \right\} \tag{2.8}$$

where

$$H(z)=\frac{zF''_{p,\mu}(z)}{F'_{p,\mu}(z)}=\frac{1}{\mu}\left[\frac{\sum_{i=1}^{k}zI^m_{\alpha,\beta,\gamma,}f'_{i,p}(t)}{\sum_{i=1}^{k}I^m_{\alpha,\beta,\gamma,}f_{i,p}(t)}-1\right]$$

and where $0\leq\xi_1,\xi_2\leq 1; 0<\delta<1; |\mu|\leq 1,\mu\in\mathbb{C}$ and $\alpha,\beta,\gamma\geq 0$ with $\alpha+\beta+\gamma>0$. The author in [14] provided the following coefficient bound:
Let $I^m_{\alpha,\beta,\gamma,}f_{i,p}(t)$ be in $\Gamma_{p,\mu,\alpha,\beta\gamma}(\xi_1,\xi_2,\delta)$ then

$$|a_n^i|\leq\frac{(\alpha+\beta+\gamma)^m(\delta(p\xi_1+\mu\xi_2)+|\mu-p|)}{(\alpha+n\beta+n^2\gamma)^m[n(1+\delta\xi_1)+\mu(\delta\xi_2-1)]}. \tag{2.9}$$

where $n\geq 2, 0\leq\xi_1,\xi_2\leq 1; 0<\delta<1; |\mu|\leq 1,\mu\in\mathbb{C},p\in\mathbb{N}$ and $\alpha,\beta,\gamma\geq 0$ with $\alpha+\beta+\gamma>0$. Furthermore, we define the following well-known existing differential operators and the absolute error:

Definition 2.1. Salagean Differential Operator [17]: Let $f(z)\in\mathbf{A}$, then the Salagean differential operator $D^m:\mathbf{A}\to\mathbf{A}$, is defined by:

$$D^mf(z)=D(D^{m-1}f(z))=z+\sum_{n=2}^{\infty}n^ma_nz^n. \tag{2.10}$$

for any $m\in\mathbb{N}$.

Definition 2.2. Al- Oboudi Differential Operator [15]: Let $f(z)\in\mathbf{A}$, then the Al-Oboudi differential operator $D_t^m:\mathbf{A}\to\mathbf{A}, m\in\mathbf{N}\bigcup\{0\}, t\geq 0$ is defined by:

$$D_t^mf(z)=D_t(D_t^{m-1}f(z))=z+\sum_{n=2}^{\infty}(1+(n-1)t)^ma_nz^n. \tag{2.11}$$

Definition 2.3. Salagean q-Differential Operator [18]: Let $f(z)\in\mathbf{A}$, then the Salagean q-differential operator $D_q^m:\mathbf{A}\to\mathbf{A}$, is defined by:

$$D_q^mf(z)=D_q(D_q^{m-1}f(z))=z+\sum_{n=2}^{\infty}[n]_q^ma_nz^n. \tag{2.12}$$

for any $m\in\mathbb{N}$.

Definition 2.4. Absolute and Relative Errors. Let P denote the exact (or reference) value of a quantity, and let $\tilde{P}$ denote its approximate or computed value. Then:

$$E_{\text{abs}}=|P-\tilde{P}|$$

is called the *absolute error*, while

$$E_{\text{rel}}=\frac{|P-\tilde{P}|}{|P|}$$

is called the *relative error*.

The absolute error provides a direct measure of the deviation between the exact and approximate results, whereas the relative error measures this deviation in proportion to the true value. In this paper, these quantities are introduced to facilitate numerical comparisons between the new operator $I^m_{q,\vartheta,\alpha,\beta,\gamma}$ and existing operators. Specifically, the relative error allows for assessing the efficiency and accuracy of the proposed operator in producing smaller coefficient bounds and more refined numerical results, as presented in Tables 1–2 and Figures 3–7.

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3 MAIN RESULTS

Using the concept of q-number and q-derivatives, we defined the new operator introduced and the class of p-valent analytic functions in definition 3.1.

Definition 3.1. *: Let $q \in (0,1), \vartheta \geq 1, \alpha, \gamma, \beta, \geq 0,$ with $\alpha + \beta\vartheta + \gamma\vartheta^2 > 0,$ then for a subclass of* **A** *of the form*

$$f_i(z) = z + \sum_{n=2}^{\infty} a_n^i z^n \tag{3.1}$$

and $1 \leq i \leq k, |z| < 1,$ we define the linear operator of the form $I_{q,\vartheta,\alpha,\beta,\gamma}$ defined by

$$I_{q,\vartheta,\alpha,\beta,\gamma} f_i(z) = \frac{\alpha f_i(z) + \beta\vartheta z D_q f_i(z) + \gamma\vartheta^2 z D_q(z D_q f_i(z))}{\alpha + \beta\vartheta + \gamma\vartheta^2} \tag{3.2}$$

$$I^m_{q,\vartheta,\alpha,\beta,\gamma} f_i(z) = I_{q,\vartheta,\alpha,\beta,\gamma}(I^{m-1}_{q,\vartheta,\alpha,\beta,\gamma} f_i(z)).$$

Using (3.1) in (3.2), we have,

$$D_q f_i(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n^i z^{n-1}$$

$$z D_q f_i(z) = z + \sum_{n=2}^{\infty} [n]_q a_n^i z^n$$

$$D_q(z D_q f_i(z)) = 1 + \sum_{n=2}^{\infty} [n]_q^2 a_n^i z^{n-1}$$

$$z D_q(z D_q f_i(z)) = z + \sum_{n=2}^{\infty} [n]_q^2 a_n^i z^n$$

then

$$I_{q,\vartheta,\alpha,\beta,\gamma} f_i(z) = \frac{1}{\alpha + \beta\vartheta + \gamma\vartheta^2}\left(\alpha\left(z + \sum_{n=2}^{\infty} a_n^i z^n\right) + \beta\vartheta\left(z + \sum_{n=2}^{\infty}[n]_q a_n^i z^n\right) + \gamma\vartheta^2\left(z + \sum_{n=2}^{\infty}[n]_q^2 a_n^i z^n\right)\right)$$

$$I_{q,\vartheta,\alpha,\beta,\gamma} f_i(z) = \frac{(\alpha + \beta\vartheta + \gamma\vartheta^2)z + \alpha\sum_{n=2}^{\infty} a_n^i z^n + \beta\vartheta\sum_{n=2}^{\infty}[n]_q a_n^i z^n + \gamma\vartheta^2\sum_{n=2}^{\infty}[n]_q^2 a_n^i z^n}{\alpha + \beta\vartheta + \gamma\vartheta^2}$$

$$I_{q,\vartheta,\alpha,\beta,\gamma} f_i(z) = z + \sum_{n=2}^{\infty}\left(\frac{\alpha + [n]_q\beta\vartheta + [n]_q^2\gamma\vartheta^2}{\alpha + \beta\vartheta + \gamma\vartheta}\right) a_n^i z^n.$$

and

$$I^2_{q,\vartheta,\alpha,\beta,\gamma} f_i(z) = I_{q,\vartheta,\alpha,\beta,\gamma}(I_{q,\vartheta,\alpha,\beta,\gamma} f_i(z))$$

$$I^2_{q,\vartheta,\alpha,\beta,\gamma} f_i(z) = z + \sum_{n=2}^{\infty}\left(\frac{\alpha + [n]_q\beta\vartheta + [n]_q^2\gamma\vartheta^2}{\alpha + \beta\vartheta + \gamma\vartheta^2}\right)^2 a_n^i z^n.$$

For any $m \in \mathbb{N}$, we have

$$I^m_{q,\vartheta,\alpha,\beta,\gamma} f_i(z) = z + \sum_{n=2}^{\infty}\left(\frac{\alpha + [n]_q\beta\vartheta + [n]_q^2\gamma\vartheta^2}{\alpha + \beta\vartheta + \gamma\vartheta^2}\right)^m a_n^i z^n.$$

Remark 3.2. : For a function of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ with $q \in (0,1), \vartheta \geq 1, \alpha, \gamma, \beta, \geq 0,$ with $\alpha + \beta\vartheta + \gamma\vartheta^2 > 0$, it follows that

i. $I_{q,\vartheta,\alpha,\beta,\gamma}$ is a linear operator.

ii. $I_{q,\vartheta,\alpha,0,0}f(z) = f(z)$

iii. $\lim_{q\to 1} I_{q,1,\alpha,\beta,\gamma}f(z) = I_{\alpha,\beta,\gamma}f(z)$ which is the Makinde et al operator [9] in (2.6).

iv. $\lim_{q\to 1} I_{q,1,\alpha,\beta,0}f(z) = I_{\alpha,\beta}f(z)$ which is the Swammy differential operator [10] in (2.5).

v. $\lim_{q\to 1} I^m_{q,1,1-\beta,\beta,0}f(z) = D^m_\beta f(z)$ where $m \in \mathbb{N}$, which is AL-Oboudi differential operator [15] in (2.11).

vi. $\lim_{q\to 1} I^m_{q,1,0,1,0}f(z) = D^m f(z)$ where $m \in \mathbb{N}$, which is the Salagean differential operator [17] in (2.10).

vii. $I^m_{q,1,0,1,0}f(z) = D^m_q f(z)$ where $m \in \mathbb{N}$, which is the Salagean q-differential operator [18] in (2.12).

Now, suppose

$$I^m_{q,\vartheta,\alpha,\beta,\gamma}f_{i,p}(z) = z^p + \sum_{n=p+1}^{\infty}\left(\frac{\alpha + [n]_q\beta\vartheta + [n]_q^2\gamma\vartheta^2}{\alpha + \beta\vartheta + \gamma\vartheta^2}\right)^m a^i_n z^n,$$

where $p \in \mathbb{N}$, such that when p=1 we have

$$I^m_{q,\vartheta,\alpha,\beta,\gamma}f_{i,1}(z) = I^m_{q,\vartheta,\alpha,\beta,\gamma}f_i(z) = z + \sum_{n=2}^{\infty}\left(\frac{\alpha + [n]_q\beta\vartheta + [n]_q^2\gamma\vartheta^2}{\alpha + \beta\vartheta + \gamma\vartheta^2}\right)^m a^i_n z^n.$$

then we define the class

$$\Gamma_{q,p,\vartheta,\mu,\alpha,\beta,\gamma}(\xi_1,\xi_2,\delta) = \left\{ I^m_{q,\vartheta,\alpha,\beta,\gamma,}f_{i,p}(t) \in A : \left|\frac{H_q(z) + \frac{1}{\mu} - 1}{\xi_1(H_q(z) + \frac{1}{\mu}) + \xi_2}\right| < \delta \right\} \tag{3.3}$$

where $0 \le \xi_1, \xi_2 \le 1; 0 < \delta < 1; \alpha, \beta, \gamma \ge 0, 0 < \mu \le 1, \vartheta \ge 1, q \in (0,1), \alpha + \beta\vartheta + \gamma\vartheta^2 > 0$ and

$$H_q(z) = \frac{1}{\mu}\left[\frac{\sum_{i=1}^{k} zI^m_{q,\vartheta,\alpha,\beta,\gamma,}f'_{i,p}(z)}{\sum_{i=1}^{k} I^m_{q,\vartheta,\alpha,\beta,\gamma,}f_{i,p}(z)} - 1\right].$$

In the following section, we obtain the coefficient bound for the class $\Gamma_{q,p,\vartheta,\mu,\alpha,\beta\gamma}(\xi_1,\xi_2,\delta)$.

Theorem 3.3. : Let $I^m_{q,\vartheta,\alpha,\beta,\gamma}f_{i,p}$ belong to the class $\Gamma_{q,p,\vartheta,\mu,\alpha,\beta\gamma}(\xi_1,\xi_2,\delta)$. if

$$\sum_{i=1}^{k}\sum_{n=p+1}^{\infty}(|n\delta\xi_1 + \delta\mu\xi_2| + |n-\mu|)\left(\frac{\alpha + [n]_q\beta\vartheta + [n]_q^2\gamma\vartheta^2}{\alpha + \beta\vartheta + \gamma\vartheta^2}\right)^m |a^i_n| \le \delta|p\xi_1 + \mu\xi_2| - |\mu - p|$$

with $|p\xi_1 + \mu\xi_2| > \sum_{i=1}^{k}\sum_{n=p+1}^{\infty}(n\xi_1 + \mu\xi_2)\left(\frac{\alpha + [n]_q\beta\vartheta + [n]_q^2\gamma\vartheta^2}{\alpha + \beta\vartheta + \gamma\vartheta^2}\right)^m |a^i_n|$ then

$$|a^i_n| \le \frac{(\alpha + \beta\vartheta + \gamma\vartheta^2)^m\delta|p\xi_1 + \mu\xi_2| + |\mu - p|}{(\alpha + [n]_q\beta\vartheta + [n]_q^2\gamma\vartheta^2)^m(|n\delta\xi_1 + \delta\mu\xi_2| + |n-\mu|)}$$

where $p, m \in \mathbb{N}, 0 \le \xi_1, \xi_2 \le 1; 0 < \delta < 1; 0 < \mu \le 1, \vartheta \ge 1, q \in (0,1), \alpha, \beta, \gamma \ge 0$ with $\alpha + \beta\vartheta + \gamma\vartheta^2 > 0$.

Proof: Let $I^m_{q,\vartheta,\alpha,\beta,\gamma,}f_{i,p}$ belong to the class $\Gamma_{q,p,\vartheta,\mu,\alpha,\beta\gamma}(\xi_1,\xi_2,\delta)$
Recall that

$$\Gamma_{q,p,\vartheta,\mu,\alpha,\beta\gamma}(\xi_1,\xi_2,\delta) = \left\{ I^m_{q,\vartheta,\alpha,\beta,\gamma,}f_{i,p}(t) \in A : |\frac{H_q(z) + \frac{1}{\mu} - 1}{\xi_1(H_q(z) + \frac{1}{\mu}) + \xi_2}| < \delta \right\}$$

where

$$H_q(z) = \frac{1}{\mu}\left(\frac{\sum_{i=1}^{k} zI^m_{q,\vartheta,\alpha,\beta,\gamma,}f'_{i,p}(z)}{\sum_{i=1}^{k} I^m_{q,\vartheta,\alpha,\beta,\gamma,}f_{i,p}(z)} - 1\right). \tag{3.4}$$

But

$$I^m_{q,\vartheta,\alpha,\beta,\gamma,}f_{i,p}(z) = z^p + \sum_{n=p+1}^{\infty}\left(\frac{\alpha + [n]_q\beta\vartheta + [n]_q^2\gamma\vartheta^2}{\alpha+\beta\vartheta+\gamma\vartheta^2}\right)^m a_n^i z^n \tag{3.5}$$

$$I^m_{q,\vartheta,\alpha,\beta,\gamma,}f'_{i,p}(z) = pz^{p-1} + \sum_{n=p+1}^{\infty}\left(\frac{\alpha + [n]_q\beta\vartheta + [n]_q^2\gamma\vartheta^2}{\alpha+\beta\vartheta+\gamma\vartheta^2}\right)^m na_n^i z^{n-1}. \tag{3.6}$$

Then, substituting (3.5) and (3.6) into (3.4), we have

$$\begin{aligned}
H_q(z) &= \frac{1}{\mu}\left(\frac{\sum_{i=1}^{k} zI^m_{q,\alpha,\beta,\gamma,}f'_{i,p}(z)}{\sum_{i=1}^{k} I^m_{q,\alpha,\beta,\gamma,}f_{i,p}(z)} - 1\right)\\
&= \frac{1}{\mu}\left(\frac{\sum_{i=1}^{k}\left(pz^p + \sum_{n=p+1}^{\infty} nX^m a_n^i z^n\right)}{\sum_{i=1}^{k}\left(z^p + \sum_{n=p+1}^{\infty} X^m a_n^i z^n\right)} - 1\right)\\
&= \frac{1}{\mu}\left(\frac{\sum_{i=1}^{k}(pz^p + \sum_{n=p+1}^{\infty} nX^m a_n^i z^n) - \sum_{i=1}^{k}(z^p + \sum_{n=p+1}^{\infty} X^m a_n^i z^n)}{\sum_{i=1}^{k}(z^p + \sum_{n=p+1}^{\infty} X^m a_n^i z^n)}\right)\\
&= \frac{1}{\mu}\left(\frac{\sum_{i=1}^{k}(p-1)z^p + \sum_{n=p+1}^{\infty}(n-1)X^m a_n^i z^n}{\sum_{i=1}^{k}(z^p + \sum_{n=p+1}^{\infty} X^m a_n^i z^n)}\right),
\end{aligned}$$

where

$$X = \left(\frac{\alpha + [n]_q\beta\vartheta + [n]_q^2\gamma\vartheta^2}{\alpha+\beta\vartheta+\gamma\vartheta^2}\right) \tag{3.7}$$

then

$$\begin{aligned}
\left|\frac{H_q(z) + \frac{1}{\mu} - 1}{\xi_1(H_q(z) + \frac{1}{\mu}) + \xi_2}\right| &= \left|\frac{\frac{\sum_{i=1}^{k}(pz^p + \sum_{n=p+1}^{\infty} nX^m a_n^i z^n - \mu\sum_{i=1}^{k}(z^p + \sum_{n=p+1}^{\infty} X^m a_n^i z^n)}{\mu\sum_{i=1}^{k}(z^p + \sum_{n=p+1}^{\infty} X^m a_n^i z^n)}}{\frac{\xi_1\sum_{i=1}^{k}(pz^p + \sum_{n=p+1}^{\infty} X^m na_n^i z^n) + \xi_2\sum_{i=1}^{k}(z^p + \sum_{n=p+1}^{\infty} X^m a_n^i z^n)}{\mu\sum_{i=1}^{k}(z^p + \sum_{n=p+1}^{\infty} X^m a_n^i z^n)}}\right|\\
&= \left|\frac{\sum_{i=1}^{k}(pz^p + \sum_{n=p+1}^{\infty} nX^m a_n^i z^n - \mu\sum_{i=1}^{k}(z^p + \sum_{n=p+1}^{\infty} X^m a_n^i z^n)}{\xi_1\sum_{i=1}^{k}(pz^p + \sum_{n=p+1}^{\infty} X^m na_n^i z^n) + \mu\xi_2\sum_{i=1}^{k}(z^p + \sum_{n=p+1}^{\infty} X^m a_n^i z^n)}\right|\\
&= \left|\frac{\sum_{i=1}^{k}((p-\mu)z^p + \sum_{n=p+1}^{\infty}(n-\mu)X^m a_n^i z^n)}{\sum_{i=1}^{k}((p\xi_1 + \mu\xi_2)z^p + \sum_{n=p+1}^{\infty}(n\xi_1 + \mu\xi_2)X^m a_n^i z^n)}\right|\\
&= \frac{|\sum_{i=1}^{k}((p-\mu)z^p + \sum_{n=p+1}^{\infty}(n-\mu)X^m a_n^i z^n)|}{|\sum_{i=1}^{k}((p\xi_1 + \mu\xi_2)z^p + \sum_{n=p+1}^{\infty}(n\xi_1 + \mu\xi_2)X^m a_n^i z^n)|}\\
&\leq \frac{\sum_{i=1}^{k}(|p-\mu||z^p| + \sum_{n=p+1}^{\infty}|n-\mu|X^m|a_n^i||z^n|)}{\sum_{i=1}^{k}(|p\xi_1 + \mu\xi_2||z^p| + \sum_{n=p+1}^{\infty}|n\xi_1 + \mu\xi_2|X^m|a_n^i||z^n|)} \qquad (3.8)
\end{aligned}$$

for $z \to 1^{-1}$ and $|p\xi_1 + \mu\xi_2| > \sum_{i=1}^{k}\sum_{n=p+1}^{\infty}(n\xi_1 + |\mu|\xi_2)X^m|a_n^i|$, we have that

$$\left|\frac{H_q(z) + \frac{1}{\mu} - 1}{\xi_1(H_q(z) + \frac{1}{\mu}) + \xi_2}\right| \leq \frac{|p-\mu| + \sum_{i=1}^{k}\sum_{n=p+1}^{\infty}|n-\mu|X^m|a_n^i|}{|p\xi_1 + \mu\xi_2| - \sum_{i=1}^{k}\sum_{n=p+1}^{\infty}|n\xi_1 + \mu\xi_2|X^m|a_n^i|} \tag{3.9}$$

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The right hand side of (3.9) is bounded by δ if

$$|p-\mu|+\sum_{i=1}^{k}\sum_{n=p+1}^{\infty}|n-\mu|X^m|a_n^i|\leq\delta|p\xi_1+\mu\xi_2|-\delta\sum_{i=1}^{k}\sum_{n=p+1}^{\infty}|n\xi_1+\mu\xi_2|X^m|a_n^i|, \tag{3.10}$$

It follows that

$$\sum_{i=1}^{k}\sum_{n=p+1}^{\infty}(|n\delta\xi_1+\delta\mu\xi_2|+|n-\mu|)X^m|a_n^i|\leq\delta|p\xi_1+\mu\xi_2|-|\mu-p|, \tag{3.11}$$

But

$$\delta|p\xi_1+\mu\xi_2|-|\mu-p|\leq\delta|p\xi_1+\mu\xi_2|+|\mu-p| \tag{3.12}$$

Then,

$$\sum_{i=1}^{k}\sum_{n=p+1}^{\infty}(|n\delta\xi_1+\delta\mu\xi_2|+|n-\mu|)X^m|a_n^i|\leq\delta|p\xi_1+\mu\xi_2|+|\mu-p|, \tag{3.13}$$

Using (3.7) in (3.13), we have

$$\sum_{i=1}^{k}\sum_{n=p+1}^{\infty}\left(\frac{\alpha+[n]_q\beta\vartheta+[n]_q^2\gamma\vartheta^2}{\alpha+\beta\vartheta+\gamma\vartheta^2}\right)^m(|n\delta\xi_1+\delta\mu\xi_2|+|n-\mu|)|a_n^i|\leq\delta|p\xi_1+\mu\xi_2|+|\mu-p| \tag{3.14}$$

Therefore, we have

$$|a_n^i|\leq\frac{(\alpha+\beta\vartheta+\gamma\vartheta^2)^m(\delta|p\xi_1+\mu\xi_2|+|\mu-p|)}{(\alpha+[n]_q\beta\vartheta+[n]_q^2\gamma\vartheta^2)^m(|n\delta\xi_1+\delta\mu\xi_2|+|n-\mu|)}. \tag{3.15}$$

Corollary 3.4. *Let all hypotheses of Theorem 1 hold with $\vartheta=1$. In the classical limit $q\to1$ (so that $[n]_q\to n$), the coefficient estimate in Theorem 1 reduces to*

$$|a_n^i|\leq\frac{(\alpha+\beta+\gamma)^m\left(\delta|p\xi_1+\mu\xi_2|+|\mu-p|\right)}{(\alpha+n\beta+n^2\gamma)^m\left(|n\delta\xi_1+\delta\mu\xi_2|+|\,n-\mu\,|\right)}.$$

In particular, this reproduces the inequality (2.9) (for suitable choices of ξ_1,ξ_2,μ).

Corollary 3.5 (Univalent Case $p=1$)**.** *If $p=1$ and all other assumptions of Theorem 1 hold, then for every $n\geq2$*

$$|a_n^i|\leq\frac{\left(\alpha+\beta\vartheta+\gamma\vartheta^2\right)^m\left(\delta|\xi_1+\mu\xi_2|+|\mu-1|\right)}{\left(\alpha+[n]_q\beta\vartheta+[n]_q^2\gamma\vartheta^2\right)^m\left(|n\delta\xi_1+\delta\mu\xi_2|+|\,n-\mu\,|\right)}.$$

4 COMPARISON OF RESULTS

In this section, we compare the existing results with the new result, highlighting their similarities and differences. Table 1 and Table 2 present the analysis of coefficients for inequalities (2.9) (Existing Results) and (3.15) (New Results), respectively, while Table 3 provides a summary of the comparison of results.

From Tables 1 and 2, we observe that the coefficients in Table 2 (New Results) are consistently smaller than those in Table 1 (Existing Results) for the same values of N and M. For example, when $N=5$ and $M=3$, the coefficient in the existing result is 1.679727×10^{-4}, while the corresponding new result is 4.648208×10^{-5}. Similarly, for $N=10$ and $M=2$, the existing coefficient is 6.707556×10^{-5} compared to 2.538631×10^{-5} in the new result. These examples illustrate that the

analysis in Table 2 provides finer bounds than that in Table 1.
Consider the inequality (2.9),

$$|a_n^i| \leq \frac{(\alpha+\beta+\gamma)^m(\delta(p\xi_1+\mu\xi_2)+|\mu-p|)}{(\alpha+n\beta+n^2\gamma)^m[n(1+\delta\xi_1)+\mu(\delta\xi_2-1)]},$$

where $n \geq 2, 0 \leq \xi_1, \xi_2 \leq 1; 0 < \delta < 1; 0 < \mu \leq 1$, and $\alpha, \beta, \gamma \geq 0$ with $\alpha+\beta+\gamma > 0, m \in \mathbb{N}$
To analyze inequality (9) and generate Table 1, the following conditions were applied: $\alpha = 1, \beta = 1, \gamma = 1, m = M, \delta = 0.1, \xi_1 = 1, \mu = 0.1, \xi_2 = 0.1, n = N, p = 1$. The behavior was studied for n ranging from 2 to 15 with m taking the values 1, 2, 3, 4, and 5.

Table 1: **Analysis of coefficients in inequality (2.9) (Existing Results) for n ranging from 2 to 15, with m taking the values of 1, 2, 3, 4, and 5**

	M=1	M=2	M=3	M=4	M=5
N=2	**0.2041885**	**0.08750935**	3.750401×10^{-2}	1.607315×10^{-2}	6.888491×10^{-3}
N=3	**0.07216495**	**0.01665345**	3.843104×10^{-3}	8.868701×10^{-4}	2.046623×10^{-4}
N=4	**0.03324808**	**0.004749726**	6.785323×10^{-4}	9.693318×10^{-5}	1.384760×10^{-5}
N=5	**0.01793575**	**0.001735717**	1.679727×10^{-4}	1.625542×10^{-5}	1.573105×10^{-6}
N=6	**0.01074253**	**0.0007494791**	5.228924×10^{-5}	3.648086×10^{-6}	2.545177×10^{-7}
N=7	**0.006931221**	**0.0003648011**	1.920006×10^{-5}	1.010529×10^{-6}	5.318576×10^{-8}
N=8	**0.004727846**	**0.000194295**	7.984727×10^{-6}	3.281395×10^{-7}	1.348518×10^{-8}
N=9	**0.003367003**	**0.0001110001**	3.659344×10^{-6}	1.206377×10^{-7}	3.977068×10^{-9}
N=10	**0.002481796**	**0.00006707556**	1.812853×10^{-6}	4.899602×10^{-8}	1.324217×10^{-9}
N=11	**0.001881422**	**0.00004243809**	9.572502×10^{-7}	2.159211×10^{-8}	4.870401×10^{-10}
N=12	**0.001459995**	**0.00002789798**	5.330825×10^{-7}	1.018629×10^{-8}	1.946425×10^{-10}
N=13	**0.001155541**	**0.00001894329**	3.105458×10^{-7}	5.090915×10^{-9}	8.345762×10^{-11}
N=14	**0.0009301502**	**0.00001322488**	1.880315×10^{-7}	2.673434×10^{-9}	3.801091×10^{-11}
N=15	**0.0007597452**	**0.000009457409**	1.177271×10^{-7}	1.465482×10^{-9}	1.824252×10^{-11}

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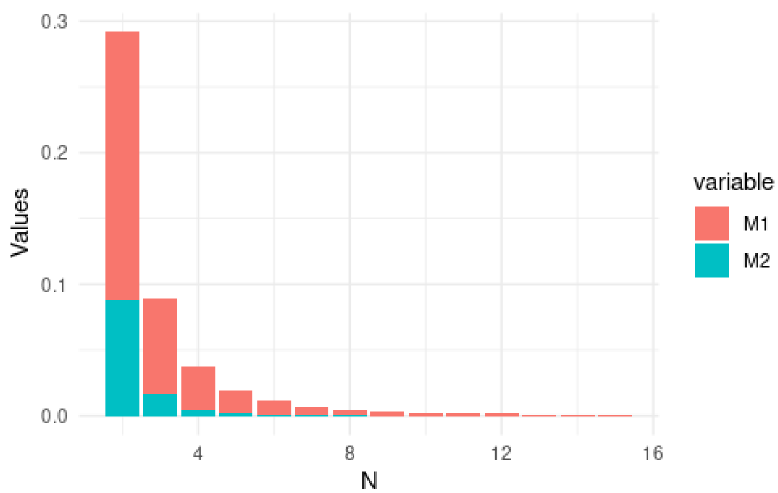

Figure 1: coefficients in inequality (2.9)

We now consider inequality (27),

$$|a_n^i| \leq \frac{(\alpha+\beta\vartheta+\gamma\vartheta^2)^m(\delta|p\xi_1+\mu\xi_2|+|\mu-p|)}{(\alpha+[n]_q\beta\vartheta+[n]_q^2\gamma\vartheta^2)^m(|n\delta\xi_1+\delta\mu\xi_2|+|n-\mu|)}$$

$m\in\mathbb{N}, n\geq 2, 0\leq\xi_1,\xi_2\leq 1; 0<\delta<1; 0<\mu\leq 1, \vartheta\geq 1, q\in(0,1), \alpha,\beta,\gamma\geq 0$ with $\alpha+\beta\vartheta+\gamma\vartheta^2>0$.
To generate table 2, we set the following conditions: $\alpha=1,\beta=1,\gamma=1,m=M,\delta=0.1,\xi_1=1,\mu=0.1,\xi_2=0.1,n=N; q=0.9999999, p=1,\vartheta=2$. Note that $\lim_{q\to 1}[n]_q=n$. The behavior was studied for n ranging from 2 to 15, with m taking the values of 1, 2, 3, 4, and 5.

Table 2: **Analysis of coefficients in inequality (3.15) (New Results) for n ranging from 2 to 15, with m taking the values of 1, 2, 3, 4, and 5**

	M=1	M=2	M=3	M=4	M=5
N=2	**0.1588133**	**0.05293775**	$\mathbf{1.764592\times 10^{-2}}$	$\mathbf{5.881975\times 10^{-3}}$	$\mathbf{1.960658\times 10^{-3}}$
N=3	**0.05090706**	**0.008287195**	$\mathbf{1.349079\times 10^{-3}}$	$\mathbf{2.196176\times 10^{-4}}$	$\mathbf{3.575170\times 10^{-5}}$
N=4	**0.02231721**	**0.00214006**	$\mathbf{2.052062\times 10^{-4}}$	$\mathbf{1.967732\times 10^{-5}}$	$\mathbf{1.886866\times 10^{-6}}$
N=5	**0.01168786**	**0.0007370722**	$\mathbf{4.648208\times 10^{-5}}$	$\mathbf{2.931304\times 10^{-6}}$	$\mathbf{1.848571\times 10^{-7}}$
N=6	**0.006865186**	**0.0003060911**	$\mathbf{1.364739\times 10^{-5}}$	$\mathbf{6.084828\times 10^{-7}}$	$\mathbf{2.712982\times 10^{-8}}$
N=7	**0.004368969**	**0.0001449421**	$\mathbf{4.808514\times 10^{-6}}$	$\mathbf{1.595242\times 10^{-7}}$	$\mathbf{5.292276\times 10^{-9}}$
N=8	**0.002949853**	**0.00007563724**	$\mathbf{1.939420\times 10^{-6}}$	$\mathbf{4.972876\times 10^{-8}}$	$\mathbf{1.275097\times 10^{-9}}$
N=9	**0.002084335**	**0.00004253746**	$\mathbf{8.681134\times 10^{-7}}$	$\mathbf{1.771661\times 10^{-8}}$	$\mathbf{3.615638\times 10^{-10}}$
N=10	**0.001526805**	**0.00002538631**	$\mathbf{4.221013\times 10^{-7}}$	$\mathbf{7.018318\times 10^{-9}}$	$\mathbf{1.166942\times 10^{-10}}$
N=11	**0.001151613**	**0.00001589999**	$\mathbf{2.195271\times 10^{-7}}$	$\mathbf{3.030949\times 10^{-9}}$	$\mathbf{4.184747\times 10^{-11}}$
N=12	**0.0008899246**	**0.00001036518**	$\mathbf{1.207263\times 10^{-7}}$	$\mathbf{1.406131\times 10^{-9}}$	$\mathbf{1.637758\times 10^{-11}}$
N=13	**0.0007018719**	**0.000006988767**	$\mathbf{6.958968\times 10^{-8}}$	$\mathbf{6.929279\times 10^{-10}}$	$\mathbf{6.899717\times 10^{-12}}$
N=14	**0.0005632767**	**0.000004849861**	$\mathbf{4.175788\times 10^{-8}}$	$\mathbf{3.595394\times 10^{-10}}$	$\mathbf{3.095669\times 10^{-12}}$
N=15	**0.0004588937**	**0.000003450329**	$\mathbf{2.594243\times 10^{-8}}$	$\mathbf{1.950561\times 10^{-10}}$	$\mathbf{1.466589\times 10^{-12}}$

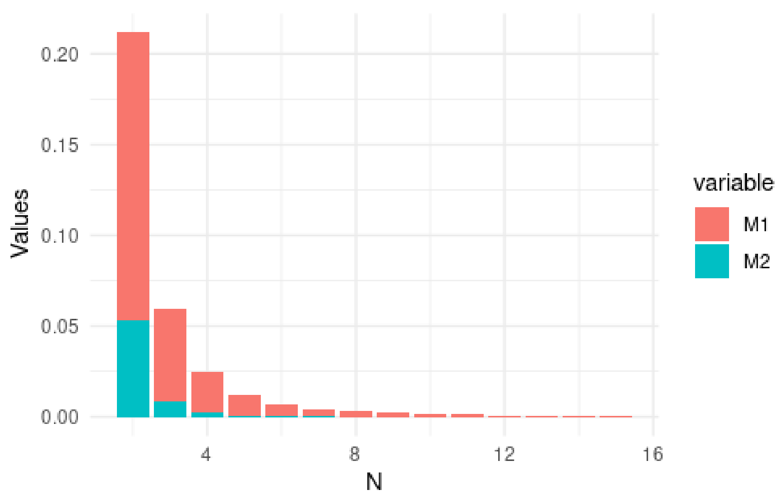

Figure 2: coefficients in inequality (3,15)

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Table 3: Comparison of New and Existing Results

S/N	New Result	Existing Result(s)	Similarity(ies) and Difference(s)
1.	$I^m_{q,\vartheta,\alpha,\beta,\gamma} f_i(z) = z + \sum_{n=2}^{\infty}\left(\frac{\alpha + [n]_q\beta\vartheta + [n]_q^2\gamma\vartheta^2}{\alpha+\beta\vartheta+\gamma\vartheta^2}\right)^m a_n^i z^n$	• Salagean Differential Operator: $D^m f(z) = z + \sum_{n=0}^{\infty} n^m a_n z^n$. • Al-Oboudi Operator: $D_t^m f(z) = z + \sum_{n=2}^{\infty}(1+(n-1)t)^m a_n z^n$. • Salagean q-Operator: $D_q^m f(z) = z + \sum_{n=0}^{\infty}[n]_q^m a_n z^n$. • Swamy : $I^m_{\alpha,\beta} f(z) = z + \sum_{n=2}^{\infty}\left(\frac{\alpha+n\beta}{\alpha+\beta}\right)^m a_n z^n$. • Makinde et al. : $I^m_{\alpha,\beta,\gamma} f(z) = z + \sum_{n=2}^{\infty}\left(\frac{\alpha+n\beta+n^2\gamma}{\alpha+\beta+\gamma}\right)^m a_n z^n$.	• They are linear operators. • $I_{q,\vartheta,\alpha,0,0} f(z) = f(z)$. • $\lim_{q\to 1} I^m_{q,1,1-\beta,\beta,0} f(z) = D^m_\beta f(z)$.(Al-Oboudi differential operator) • $\lim_{q\to 1} I^m_{q,1,0,1,0} f(z) = D^m f(z)$.(Salagean differential operator) • $I^m_{q,1,0,1,0} f(z) = D^m_q f(z)$.(Salagean q-differential operator) • $\lim_{q\to 1} I_{q,1,\alpha,\beta,0} f(z) = I_{\alpha,\beta} f(z)$. (swamy operator) • $\lim_{q\to 1} I_{q,1,\alpha,\beta,\gamma} f(z) = I_{\alpha,\beta,\gamma} f(z)$. (Makinde et al operator)
2.	$\lvert a_n^i\rvert \le \frac{(\alpha+\beta\vartheta+\gamma\vartheta^2)^m(\delta\lvert p\xi_1+\mu\xi_2\rvert + \lvert\mu-p\rvert)}{(\alpha+[n]_q\beta\vartheta+[n]_q^2\gamma\vartheta^2)^m(\lvert n\delta\xi_1+\delta\mu\xi_2\rvert + \lvert n-\mu\rvert)}$, where $n \ge 2, 0 \le \xi_1, \xi_2 \le 1; 0 < \delta < 1; 0 < \mu \le 1, \vartheta \ge 1, q \in (0,1), \alpha, \beta, \gamma \ge 0$ with $\alpha + \beta\vartheta + \gamma\vartheta^2 > 0$.	$\lvert a_n^i\rvert \le \frac{(\alpha+\beta+\gamma)^m(\delta(p\xi_1+\mu\xi_2) + \lvert\mu-p\rvert)}{(\alpha+n\beta+n^2\gamma)^m[n(1+\delta\xi_1)+\mu(\delta\xi_2-1)]}$. where $n \ge 2, 0 \le \xi_1, \xi_2 \le 1; 0 < \delta < 1; \lvert\mu\rvert \le 1, \mu \in \mathbb{C}, p \in \mathbb{N}$ and $\alpha, \beta, \gamma \ge 0$ with $\alpha+\beta+\gamma > 0$.	The new result provides finer outcome than the existing result as n = N increases from 2 and m=M, taking the values of 1, 2, 3, 4, and 5 with q very close to 1, as shown in table 1 (Existing Results) and table 2 (New Results).

N	M_1 (ER)	M_1 (NR)	Absolute Error (Ea)	Relative Error (Er)
2	0.2041885	0.1588133	4.53752×10^{-2}	22.2221134
3	0.07216495	0.0509706	2.125789×10^{-2}	29.4357632
4	0.03324808	0.02231721	1.093887×10^{-2}	32.87669454
5	0.01793753	0.01168786	6.247894×10^{-3}	34.81618434
6	0.01074523	0.00686156	3.877344×10^{-3}	36.0933969
7	0.00683921	0.004262699	2.576511×10^{-3}	37.67229594
8	0.004272846	0.00249853	1.772996×10^{-3}	37.6068293
9	0.002788896	0.00156935	1.219545×10^{-3}	38.4662983
10	0.002048179	0.001256805	7.91409×10^{-4}	38.7984375
11	0.001481796	0.000912689	5.69014×10^{-4}	38.47828476
12	0.001099455	0.000689246	4.10209×10^{-4}	39.4607539
13	0.0008319541	0.000517979	3.14004×10^{-4}	39.2442052
14	0.0006391502	0.000425367	2.13803×10^{-4}	39.4239328
15	0.0007597452	0.000458937	3.008515×10^{-4}	39.5899997

Table 4: Analysis of coefficients in the existing results and the new results when M= 1.

$$NM_1M_1M_1$$

Figure 3: Comparison of the existing results and the new results when M = 1.

N	M_2 (ER)	M_2 (NR)	Absolute Error (Ea)	Relative Error (Er)
2	0.2041885	0.1588133	4.53752×10^{-2}	22.2221134
3	0.07216495	0.0509706	2.125789×10^{-2}	29.4357632
4	0.03324808	0.02231721	1.093887×10^{-2}	32.87669454
5	0.01793753	0.01168786	6.247894×10^{-3}	34.81618434
6	0.01074523	0.00686156	3.877344×10^{-3}	36.0933969
7	0.00683921	0.004262699	2.576511×10^{-3}	37.67229594
8	0.004272846	0.00249853	1.772996×10^{-3}	37.6068293
9	0.002788896	0.00156935	1.219545×10^{-3}	38.4662983
10	0.002048179	0.001256805	7.91409×10^{-4}	38.7984375
11	0.001481796	0.000912689	5.69014×10^{-4}	38.47828476
12	0.001099455	0.000689246	4.10209×10^{-4}	39.4607539
13	0.0008319541	0.000517979	3.14004×10^{-4}	39.2442052
14	0.0006391502	0.000425367	2.13803×10^{-4}	39.4239328
15	0.0007597452	0.000458937	3.008515×10^{-4}	39.5899997

Table 5: Analysis of coefficients in the existing results and the new results when M = 2.

$NM_2M_2M_2$

Figure 4: Comparison of the existing results and the new results when M = 2.

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n	M_3(ER)	M_3(NR)	Absolute Error (Ea)	Relative Error (Er)
2	3.750401×10^{-2}	1.764592×10^{-2}	0.01985809	52.94924463
3	3.843104×10^{-3}	1.349079×10^{-3}	0.002494025	64.89611002
4	6.785323×10^{-4}	2.052062×10^{-4}	0.000473326	69.75734243
5	1.679727×10^{-4}	4.648208×10^{-5}	0.000121491	72.32759847
6	5.228924×10^{-5}	1.364739×10^{-5}	3.86419×10^{-5}	73.90019438
7	1.920006×10^{-5}	4.808514×10^{-6}	1.43915×10^{-5}	74.95573451
8	7.984727×10^{-6}	1.939420×10^{-6}	6.04531×10^{-6}	75.71087903
9	3.659344×10^{-6}	8.681134×10^{-7}	2.79123×10^{-6}	76.27680262
10	1.812853×10^{-6}	4.221013×10^{-7}	1.39075×10^{-6}	76.71618714
11	9.572502×10^{-7}	2.195271×10^{-7}	7.37723×10^{-7}	77.06690477
12	5.330825×10^{-7}	1.207263×10^{-7}	4.12356×10^{-7}	77.35316766
13	3.105458×10^{-7}	6.958968×10^{-8}	2.40956×10^{-7}	77.59117013
14	1.880315×10^{-7}	4.175788×10^{-8}	1.46274×10^{-7}	77.79208271
15	1.465482×10^{-7}	2.594243×10^{-8}	1.20606×10^{-7}	82.2976809

Table 6: Analysis of coefficients of the existing results (ER) and the new results (NR) when M = 3.

$M_3M_3M_3$

Figure 5: Comparison of the existing results (ER) and the new results (NR) when M= 3.

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	M_4(ER)	M_4(NR)	Absolute Error(Ea)	Relative Error(Er)
n=2	1.607315×10^{-2}	5.881975×10^{-3}	0.010191175	63.40496418
n=3	8.868701×10^{-4}	2.196176×10^{-4}	0.000667253	75.23677932
n=4	9.693318×10^{-5}	1.967732×10^{-5}	7.72559×10^{-5}	79.7001192
n=5	1.625542×10^{-5}	2.931304×10^{-6}	$1.33241\ 10^{-5}$	81.96722078
n=6	3.648086×10^{-6}	6.084828×10^{-7}	3.0396×10^{-6}	83.3204919
n=7	1.010529×10^{-6}	1.595242×10^{-7}	8.51005×10^{-7}	84.21379297
n=8	3.281395×10^{-7}	4.972876×10^{-8}	2.78411×10^{-7}	84.84523808
n=9	1.206377×10^{-7}	1.771661×10^{-8}	1.02921×10^{-7}	85.31420112
n=10	4.899602×10^{-8}	7.018318×10^{-9}	4.19777×10^{-8}	85.67573856
n=11	2.159211×10^{-8}	3.030949×10^{-9}	1.85612×10^{-8}	85.96270119
n=12	1.018629×10^{-8}	1.406131×10^{-9}	8.78016×10^{-9}	86.19584756
n=13	5.090915×10^{-9}	6.929279×10^{-10}	4.39799×10^{-9}	86.38893205
n=14	2.673434×10^{-9}	3.595394×10^{-10}	2.31389×10^{-9}	86.55140168
n=15	1.465482×10^{-9}	1.950561×10^{-10}	1.27043×10^{-9}	86.68996958

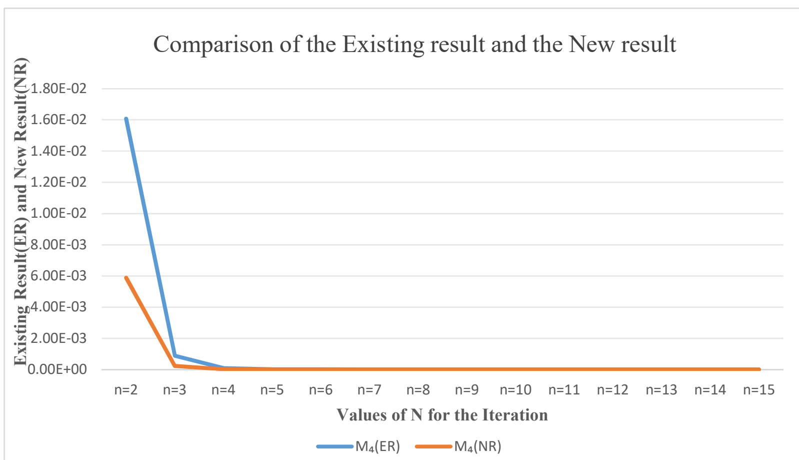

Figure 6: **Comparison of the existing results and the new results when m = 4**

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	M_5(ER)	M_5(NR)	Absolute Error(Ea)	Relative Error(Er)
n=2	6.888491 x 10^{-3}	1.960658 x 10^{-3}	0.004927833	71.53719153
n=3	2.046623 x 10^{-4}	3.575170 x 10^{-5}	0.000168911	82.53136997
n=4	1.384760 x 10^{-5}	1.886866 x 10^{-6}	1.19607 x 10^{-5}	86.3740576
n=5	1.573105 x 10^{-6}	1.848571 x 10^{-7}	1.38825 x 10^{-6}	88.24890265
n=6	2.545177 x 10^{-7}	2.712982 x 10^{-8}	2.27388 x 10^{-8}	89.34069418
n=7	5.318576 x 10^{-8}	5.292276 x 10^{-9}	4.78935 x 10^{-8}	90.04944933
n=8	1.348518 x 10^{-8}	1.275097 x 10^{-9}	1.22101 x 10^{-8}	90.54445695
n=9	3.977068 x 10^{-9}	3.615638 x 10^{-10}	3.6155 x 10^{-9}	90.90878506
n=10	1.324217 x 10^{-9}	1.166942 x 10^{-10}	1.20752 x 10^{-9}	91.18768299
n=11	4.870401 x 10^{-10}	4.184747 x 10^{-11}	4.45193 x 10^{-10}	91.40779784
n=12	1.946425 x 10^{-10}	1.637758 x 10^{-11}	1.78265 x 10^{-10}	91.58581502
n=13	8.345762 x 10^{-11}	6.899717 x 10^{-12}	7.65579 x 10^{-11}	91.73266983
n=14	3.801091 x 10^{-11}	3.095669 x 10^{-12}	3.49152 x 10^{-11}	91.85584086
n=15	1.824252 x 10^{-11}	1.466589 x 10^{-12}	1.67759 x 10^{-11}	91.96060084

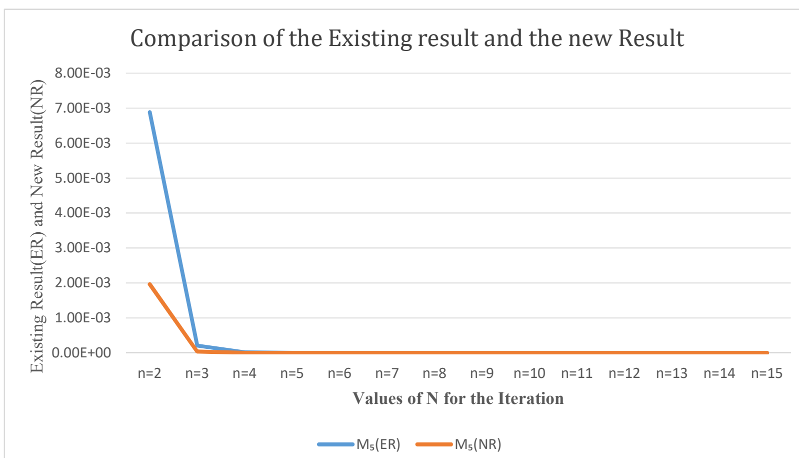

Figure 7: **Comparison of the existing results and the new results when m = 5**

5 CONCLUSION

This study introduces a generalized differential operator $I^m_{q,\vartheta,\alpha,\beta,\gamma}$ that synthesizes q-calculus, p-valency, and multiplier transformations. By incorporating parameters $\alpha, \beta, \gamma, \vartheta$, and q, the operator subsumes classical frameworks such as the Salagean, Al-Oboudi, and Makinde operators as special cases. The associated class $\Gamma_{q,p,\vartheta,\mu,\alpha,\beta\gamma}(\xi_1, \xi_2, \delta)$ of p-valent analytic functions exhibits refined coefficient bounds, validated through rigorous inequalities and numerical comparisons.
Tables 1-2 and Figures 3-7 highlight the operator's effectiveness over existing result, particularly in suppressing higher-order coefficients as n increases. For example, at $m = 5$ and $n = 15$, the new bound (Table 2) is approximately 1.8×10^{-11}, outperforming the existing result. These findings underscore the operator's precision and flexibility, making it a robust tool for analyzing analytic function subclasses.
Future work could explore applications in geometric mapping properties, radius problems, or subordination-preserving integrals. By bridging the q-calculus with traditional operator theory, this framework opens avenues for deeper interdisciplinary exploration in complex analysis.

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