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An Algorithm for Approximation of Solutions of Nonlinear Split Equality Mixed Problems

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Abstract

In this paper, we construct an iterative algorithm with a step-size which is independent of the norm of the operators that approximates a common fixed point in: the set of solutions of SEFPP involving η-demimetric maps, the set of common zeros of finite families of inverse strongly monotone maps, the set of common solutions of systems of generalized mixed equilibrium problems, and the set of common fixed points of infinite families of quasi-nonexpansive maps. We establish in real Hilbert spaces, strong convergence of the sequence generated by our algorithm to a solution of the problem under consideration.

Keywords: Quasi-nonexpansive maps, Split equality Fixed point problem, η-demimetric mappings, Strong convergence, Iterative algorithm.
MSC2010: 47H10, 47J25.

1 INTRODUCTION

Let D_1 and D_2 be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. The split feasibility problem is formulated as finding a point x satisfying

$$x \in D_1 \text{ such that } Ax \in D_2, \tag{1.1}$$

where A is bounded linear operator from H_I into H_2 . The split feasibility problem in finite dimensional Hilbert spaces was first studied by Censor and Elfving [1] for modeling inverse problems which arise in medical image reconstruction, image restoration and radiation therapy treatment planning (see e.g Byrne [2],Censor et.al [3], [1]), It is clear that $x \in D_1$ is a solution of the split feasibility problem (1.1) if and only if $Ax - P_{D_2}Ax = 0$, where P_{D_2} is the metric projection from H_2 onto D_2.

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Let H_1, H_2 and H be Hilbert spaces , let D_1, D_2 be nonempty closed convex subsets of H_1, H_2 respectively and let $A : H_1 \to H$ and $B : H_2 \to H$ be bounded linear operators. The Split Equality Problem (SEP) is to find

$$x \in D_1 \text{ and } y \in D_2 \text{ such that } Ax = By. \tag{1.2}$$

This problem has been studied by several researchers see for example Censor and Segal [3], Moudafi [4], Zhao [5] and references therein. Clearly the SEP is a special case of the SFP. Since the introduction of the Split Feasibility Problem above, many authors have modified it to solve common fixed point problems, see for example, Takahashi [6], Shehu *et al* [7], Zhao *et al* [8], Ofoedu and Araka [9], Nnubia *et al* [10] and references therein.

Ofoedu and Araka [9] considered the problem of finding common points in: the set of solutions of SEFPP involving η-demimetric maps, the set of common fixed points of a finite family of quasi-nonexpansive mappings and the set of common solutions of a finite family of classical equilibrium problems. To this end, the authors proposed the following algorithm:

$$\begin{cases} x_0, u \in H_1, \quad y_0, v \in H_2, \\ v_n = \beta x_n + (1-\beta)T_{[n]}x_n, \quad v'_n = \beta x'_n + (1-\beta)S_{[n]}x'_n, \\ w_n = \beta v_n + (1-\beta)G^{f_{[n]}}_{r_{[n]}}v_n, \quad w'_n = \beta v'_n + (1-\beta)G^{g_{[n]}}_{r_{[n]}}v'_n, \\ d_n = w_n - t_nA^*J_E(Aw_n - A'w'_n), \quad d'_n = w'_n - t_nA'^*J_E(A'w'_n - Aw_n) \\ e_n = d_n - t_n(d_n - Ud_n), \quad e'_n = d'_n - t_n(d'_n - Vd'_n), \\ x_{n+1} = \alpha_n u + (1-\alpha_n)e_n, \quad y_{n+1} = \alpha_n v + (1-\alpha_n)e'_n, \end{cases} \tag{1.3}$$

where U and V are η-demimetric maps, A and A' are bounded linear maps, $\{T_i\}_{i=1}^r$ and $\{S_i\}_{i=1}^l$ are finite families of uniformly continuous quasi-nonexpansive maps, $\{f_i\}_{i=1}^t$ and $\{g_i\}_{i=1}^p$ are finite families of bifunctions satisfying appropriate conditions, $\beta \in (0,1)$ and $\{\alpha_n\} \subset (0,1)$ satisfying appropriate conditions. Assuming the step size t_n is such that $0 < \delta < t_n \leq \gamma < \min\left\{\frac{2}{\|A\|^2+\|A'\|^2}, (1-\eta)\right\}$ for some $\gamma, \delta > 0$, and for each $i, (I - T_i), (I - S_i)$, $(I - U)$ and $(I - V)$ are demi-closed at zero, the authors proved that the sequence generated by algorithm (1.3) converges strongly to a solution of the problem they considered.

In this work, apart from the step size being independent of the operators norm, our algorithm and theorems complement and give some kind of flexibility.

2 Preliminaries

Definition 2.1. *A mapping $T : K \to K$ is said to be quasi-nonexpansive if:*
$\|Tx - p\| \leq \|x - p\|$
for all $x \in K$ and $p \in F(T)$, where $F(T)$ is the set of fixed points of T , that is ,
$F(T) = \{\ x \in K : Tx = x\ \}$.

A mapping $T : K \to K$ is said to be c-inverse strongly monotone if

$$\langle Tx - Ty, x - y\rangle \quad \geq \quad c\|Tx - Ty\|^2$$

Definition 2.2. *Let $T : K \longrightarrow K$ be a mapping and I be the identity mapping of K, we say that $(I - T)$ is demiclose at zero if and only if for any sequence $\{x_n\}_{n\geq 1}$ in K such that x_n converges weakly to x and $x_n - Tx_n \to 0$, as $n \to \infty$, we have that $x = Tx$.*

Definition 2.3. *Let K be a nonempty, closed and convex subset of a smooth Banach space E and let η and p be real numbers such that $\eta \in (-\infty, 1)$ and $1 < p < \infty$. A map $T : K \longrightarrow E$ with $F(T) \neq \emptyset$ is called η-demimetric if , for any $x \in K$ and $x^* \in Fix(T)$, we have*

$$\langle z - x^*, J_E^P(x - Tx)\rangle \geq \frac{1-\eta}{2} \parallel x - Tx \parallel^p \tag{2.1}$$

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The following lemmas will be used in the sequel.

Lemma 2.4. *(see Corollary 2.6 of Alber [11] Let $T : D(T) \subset H \to R(T) \subset H$ be a nonexpansive map with nonempty fixed point set, then $(I - T)$ is demi-closedd at zero.*

Lemma 2.5. *See eg Chidume [12] Let $x, y \in H$ and $\beta \in [0,1]$, then*

$$\|\beta x + (1-\beta)y\|^2 = \beta\|x\|^2 + (1-\beta)\|y\|^2 - \beta(1-\beta)\|x-y\|. \tag{2.2}$$

Lemma 2.6. *Let $x, y, z \in H$, then*

$$\|x-y\|^2 \leq \|x-y+z\|^2 - 2\langle z, x-y\rangle. \tag{2.3}$$

Lemma 2.7. *[12] For all $x, y \in H$, the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle. \tag{2.4}$$

Lemma 2.8. *Xu [13] Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers satisfying the following relation*

$$a_{n+1} \leq (1-\sigma_n)\, a_n + \sigma_n b_n, \quad n \geq 0, \tag{2.5}$$

where $\{\sigma_n\}_{n=0}^{\infty} \subset (0,1)$ and $\{b_n\}_{n=0}^{\infty} \subset \mathbb{R}$ satisfy the following conditions: $\sum\limits_{n=1}^{\infty} \sigma_n = \infty$ and $\limsup_{n\to\infty} b_n \leq 0$. Then, $\lim\limits_{n\to\infty} a_n = 0$.

Lemma 2.9 (Mainge [14])**.** *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then, there exists an increasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$;*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k < a_{m_k+1}.$$

In fact, m_k is the largest number n in the set $\{1, 2, ..., k\}$ such that the conditions $a_n < a_{n+1}$ holds.

For solving the generalized mixed equilibrium problem, we assume that the bifunction $f : C\times C \to \mathbb{R}$ satisfy the following properties:

$(B1)$ $f(u,u) = 0, \ \forall u \in C$,

$(B2)$ f is *monotone*, i.e., $f(u,y) + f(y,u) \leq 0, \ \forall u, y \in C$,

$(B3)$ for all $u, y, z \in C$, $\limsup_{t\downarrow 0} f(tz + (1-t)u, y) \leq f(u,y)$,

$(B4)$ for all $u \in C$, $y \longmapsto f(u,y)$ is convex and lower semicontinuous.

The following lemma can be obtain from the result of Zhang [15].

Lemma 2.10. *Let D be a nonempty closed convex subset of a real Hilbert space H and let $A : D \to H$ be a continuous and monotone mapping, $\zeta : D \to \mathbb{R}$ be a lower semi-continuous and convex function, and $f : D \times D \to \mathbb{R}$ be a bifunction satisfying the conditions $(B1) - (B4)$. Let $r > 0$ be any given number and $u \in H$ be any given point. Then, the following hold:*

1. *There exists $z \in D$ such that*

$$f(z,y) + \zeta(y) - \zeta(z) + \langle y - z, Az\rangle + \frac{1}{r}\langle y - z, z - u\rangle \geq 0, \quad \forall \ \ y \in D.$$

2. *If we define a mapping $G_r : D \to D$ by*

$$G_r(u) = \Big\{z \in D : f(z,y) + \zeta(y) - \zeta(z) + \langle y - z, Az\rangle + \frac{1}{r}\langle y-z, z-u\rangle \geq 0, \quad \forall \ \ y \in D\Big\}, \ u \in D,$$

then the mapping G_r has the following properties:

(a) G_r *is single-valued;*

(b) G_r *is a firmly nonexpansive-type mapping, that is, for all* $u, y \in H$,

$$\|G_r u - G_r y\|^2 \leq \langle G_r u - G_r y, u - y\rangle;$$

(c) $F(G_r)$ =$GMEP(f, \zeta, A) = \hat{F}(G_r)$, *where* $\hat{F}(G_r)$ *is the set of asymptotic fixed points of* G_r;

(d) $GMEP(f, \zeta, A)$ *is a closed convex subset of* D;

The following lemma can be obtained from Theorem 4.2 of the result of Nilsrakoo and Saejung [16].

Lemma 2.11. *Let* C *be a closed and convex subset of a Hilbert space,* H *and let* $\{T_i : C \to H\}_{i=0}^{\infty}$ *be a sequence of quasi-nonexpansive maps such that* $\cap_{i=0}^{\infty} F(T_i) \neq \emptyset$*. Suppose that* $\{\alpha_i\}_{i=0}^{\infty}$ *is a sequence in* $(0, 1)$ *such that* $\sum\limits_{i=0}^{\infty} \alpha_i = 1$ *and* $T : C \to H$ *is defined by*

$$Tx = \sum_{i=0}^{\infty} \alpha_i T_i x, \quad \textit{for each} \quad x \in C. \tag{2.6}$$

Then T *is well defined and the following assertions hold true.*

1. T *is quasi-nonexpansive;*
2. $F(T) = \cap_{i=0}^{\infty} F(T_i)$,
3. $I - T$ *is demi-closed at zero if and only if each* T_i *is demi-closed at zero.*

3 Main Results

We now have the following Algorithms.
Algorithm 1. Let H_1, H_2 be real Hilbert spaces and let $x_1, u \in H_1$, $y_1 v \in H_2$ be arbitrary. We generate the sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ in $H_1 \times H_2$ as follows: Given the nth term, we generate the $(n+1)th$ term by
$x_{n+1} = \alpha_n u + (1 - \alpha_n)e_n, \quad y_{n+1} = \alpha_n v + (1 - \alpha_n)e'_n,$
where
$e_n = d_n - t_n(d_n - Ud_n), \quad e'_n = d'_n - t_n(d'_n - Vd'_n);$

$d_n = w_n - t_n B^* J_E(Bw_n - B'w'_n), \quad d'_n = w'_n - t_n B'^* J_E(B'w'_n - Bw_n);$

$w_n = \beta v_n + (1 - \beta)p_n, \quad p_n = G_{r_{[n]}}^{f_{[n]}} v_n, \quad w'_n = \beta v'_n + (1 - \beta)p'_n, \quad p'_n = G_{r_{[n]}}^{g_{[n]}} v'_n;$

$v_n = \beta u_n + (1 - \beta)Tu_n, \quad v'_n = \beta u'_n + (1 - \beta)Su'_n;$

$u_n = x_n - \delta_n A_{[n]} x_n, \quad u'_n = y_n - \delta_n A'_{[n]} y_n,$
where $[n] := n \mod N$, for $n \in \mathbb{N}$ and some fixed $N \in \mathbb{N}$.

Assumptions: In what follows, we assume that $\beta \in (0, 1)$ and $\{\alpha_n\} \subset (0, 1)$ is such that: (i) $\lim\limits_{n\to\infty} \alpha_n = 0$; (ii) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$. We also have the following assumptions.

$\Omega_1 = \{(x, y) \in H_1 \times H_2 : (x, y) \in F(T) \times F(S)\}$, $\Omega_2 = \{(x, y) \in H_1 \times H_2 : (x, y) \in \left(\cap_{i=1}^{\tau} (U_i)\right) \times \left(\cap_{i=1}^{\zeta} F(V_i)\right)$ and $Bx = B'y\}$,

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$\Omega_3 = \{(x, y) \in H_1 \times H_2 : (x, y) \in \big(\cap_{i=1}^{p} A_i^{-1}(0)\big) \times \big(\cap_{i=1}^{q} A_i'^{-1}(0)\big)\}$,

$\Omega_4 = \{(x, y) \in H_1 \times H_2 : (x, y) \in \big(\cap_{i=1}^{l} GMEP(f_i, \zeta_i, \nabla_i)\big) \times \big(\cap_{i=1}^{m} GMEP(g_i, \zeta_i', \nabla_i')\big)\}$, and $\Omega = \cap_{i=1}^{4} \Omega_i$.

Theorem 3.1. *Let H_1 and H_2 be real Hilbert spaces and let X be a smooth real Banach space with dual space X^*. Let $U : H_1 \to H_1$ and $V : H_2 \to H_2$ be demimetric maps with constants η_1 and η_2 in $(-\infty, 1)$, respectively. Let $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be quasi-nonexpansive maps and let $A_d : H_1 \to H_1,\ d = 1, 2, ..., p$ and $A_h' : H_2 \to H_2,\ h = 1, 2, ..., q$ be finite families of inverse strongly monotone maps with constants $\gamma_d > 0$ and $\gamma_h > 0$, respectively. Let $\zeta_a : H_1 \to \mathbb{R}$ and $\zeta_b' : H_2 \to \mathbb{R}$ be finite families of convex and lower semi-continuous functions, $\nabla_a : H_1 \to H_1$ and $\nabla_b' : H_2 \to H_2$ be finite families of continuous monotone maps, $f_a : H_1 \times H_1 \to \mathbb{R}$ and $g_b : H_2 \times H_2 \to \mathbb{R}$ be finite families of bifunctions satisfying $(B1) - (B4)$, $\ a = 1, 2, ..., l, \ \ b = 1, 2, ..., m$. Let $B : H_1 \to X$ and $B' : H_2 \to X$ be bounded linear maps with adjoints B^* and B'^*, respectively. Let $\{(x_n, y_n)\}_{n=1}^{\infty}$ be a sequence in $H_1 \times H_2$ generated by Algorithm 1. Suppose Ω is nonempty and for some $\delta > 0$, t_n is such that*

$$\delta < t_n \leq \min\Big\{\rho, \frac{\|B'w_n' - Bw_n\|^2}{\|B^* J_X(Bw_n - B'w_n')\|^2 + \|B'^* J_X(Bw_n - B'w_n')\|^2}\Big\},\ \forall n \in \Gamma,$$

where $\Gamma := \{n \in \mathbb{N} : B'w_n' \neq Bw_n\}$; otherwise take $t_n = \min\{t, \rho\}$ (t is any positive number), where $\rho = \min\{1 - \eta, 2\gamma\}$, $\gamma = \min\{\gamma_1, \gamma_2\}$, $\gamma_1 = \min\{\gamma_d : d = 1, 2, ..., p\}$, $\gamma_2 = \min\{\gamma_h : h = 1, 2, ..., q\}$, $\eta = \max\{\eta_1, \eta_2\}$ and $\delta_n > 0$ is such that $\delta_n < 2\gamma$, then $\{(x_n, y_n)\}_{n=1}^{\infty}$ is bounded.

Proof. Let $(x, y) \in \Omega$, then by convexity of $\|.\|^2$, we have that:

$$\begin{aligned} \|x_{n+1} - x\|^2 &= \|\alpha_n(u - x) + (1 - \alpha_n)(e_n - x)\|^2 \\ &\leq \alpha_n\|u - x\|^2 + (1 - \alpha_n)\|e_n - x\|^2. \end{aligned} \tag{3.1}$$

Similarly, we obtain that:

$$\|y_{n+1} - y\|^2 \leq \alpha_n\|v - y\|^2 + (1 - \alpha_n)\|e_n' - y\|^2. \tag{3.2}$$

Adding inequality (3.1) and inequality (3.2), we have that:

$$\|x_{n+1} - x\|^2 + \|y_{n+1} - y\|^2 \leq \alpha_n(\|u - x\|^2 + \|v - y\|^2) + (1 - \alpha_n)(\|e_n' - y\|^2 + \|e_n - x\|^2). \tag{3.3}$$

Now, using the condition on η and the fact that for each $i = 1, 2, ..., t$, U_i is η_i-demimetric, we have that:

$$\begin{aligned} \|e_n - x\|^2 &= \|(d_n - x) - t_n(d_n - Ud_n)\|^2 \\ &= \|d_n - x\|^2 - 2t_n\langle d_n - Ud_n, d_n - x\rangle + t_n^2\|d_n - Ud_n\|^2 \\ &\leq \|d_n - x\|^2 - t_n(1 - \eta_{[n]})\|d_n - Ud_n\|^2 + t_n^2\|d_n - Ud_n\|^2 \\ &\leq \|d_n - x\|^2 - t_n(1 - \eta - t_n)\|d_n - Ud_n\|^2 \\ &= \|(w_n - x) - t_nB^*J_X(Bw_n - B'w_n')\|^2 - t_n(1 - \eta - t_n)\|d_n - Ud_n\|^2 \\ &= \|w_n - x\|^2 - 2t_n\langle Bw_n - Bx, J_X(Bw_n - B'w_n')\rangle \\ &\quad + t_n^2\|B^*J_X(Bw_n - B'w_n')\|^2 - t_n(1 - \eta - t_n)\|d_n - Ud_n\|^2. \end{aligned} \tag{3.4}$$

Similarly, we obtain that:

$$\begin{aligned} \|e_n' - y\|^2 &\leq \|w_n' - y\|^2 + 2t_n\langle B'w_n' - B'y, J_X(Bw_n - B'w_n')\rangle \\ &\quad + t_n^2\|B'^* J_X(Bw_n - B'w_n')\|^2 - t_n(1 - \eta - t_n)\|d_n' - Vd_n'\|^2. \end{aligned} \tag{3.5}$$

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Adding inequality (3.4) and inequality (3.5), using the fact that $Bx = B'y$ and the condition on t_n, we have that:

$$\begin{aligned} \|e_n - x\|^2 + \|e_n' - y\|^2 &\leq \|w_n - x\|^2 + \|w_n' - y\|^2 - 2t_n\|Bw_n - B'w_n'\|^2 \\ &\quad + t_n^2\big(\|B^* J_X(Bw_n - B'w_n')\|^2 + \|B'^* J_X(Bw_n - B'w_n')\|^2\big) \\ &\quad - t_n(1-\eta-t_n)(\|d_n' - Vd_n'\|^2 + \|d_n' - Vd_n'\|^2) \\ &\leq \|w_n - x\|^2 + \|w_n' - y\|^2 - t_n\|Bw_n - B'w_n'\|^2 \\ &\quad - t_n(1-\eta-t_n)\big(\|d_n - Ud_n\|^2 + \|d_n' - Vd_n'\|^2\big). \end{aligned} \tag{3.6}$$

Substituting inequality (3.6) in inequality (3.3), we have that:

$$\begin{aligned} \|x_{n+1} - x\|^2 + \|y_{n+1} - y\|^2 &\leq \alpha_n(\|u - x\|^2 + \|v - y\|^2) + (1-\alpha_n)(\|w_n - x\|^2 + \|w_n' - y\|^2) \\ &\quad - (1-\alpha_n)t_n\|Bw_n - B'w_n'\|^2 - (1-\alpha_n)t_n(1-\eta-t_n)\big(\|d_n - Ud_n\|^2 \\ &\quad + \|d_n' - Vd_n'\|^2\big). \end{aligned} \tag{3.7}$$

Using Lemma 2.5, Lemma 2.10 (e), quasi-nonexpansiveness of T, γ_d-inverse strongly monotonicity of A_d and the condition on γ, we obtain that:

$$\begin{aligned} \|w_n - x\|^2 &= \|\beta(v_n - x) + (1-\beta)(p_n - x)\|^2 \\ &= \beta\|v_n - x\|^2 + (1-\beta)\|p_n - x\|^2 - \beta(1-\beta)\|v_n - p_n\|^2 \\ &\leq \|v_n - x\|^2 - \beta(1-\beta)\|v_n - p_n\|^2 \\ &= \|\beta(u_n - x) + (1-\beta)(Tu_n - x)\|^2 - \beta(1-\beta)\|v_n - p_n\|^2 \\ &= \beta\|u_n - x\|^2 + (1-\beta)\|Tu_n - x\|^2 \\ &\quad - \beta(1-\beta)\|u_n - Tu_n\|^2 - \beta(1-\beta)\|v_n - p_n\|^2 \\ &\leq \|u_n - x\|^2 - \beta(1-\beta)\big(\|u_n - Tu_n\|^2 + \|v_n - p_n\|^2\big) \\ &= \|(x_n - x) - \delta_n A_{[n]}x_n\|^2 - \beta(1-\beta)\big(\|u_n - Tu_n\|^2 + \|v_n - p_n\|^2\big) \\ &= \|x_n - x\|^2 - 2\delta_n\langle A_{[n]}x_n, x_n - x\rangle + \delta_n^2\|A_{[n]}x_n\|^2 \\ &\quad - \beta(1-\beta)\big(\|u_n - Tu_n\|^2 + \|v_n - p_n\|^2\big) \\ &\leq \|x_n - x\|^2 - \delta_n(2\gamma - \delta_n)\|A_{[n]}x_n\|^2 - \beta(1-\beta)\big(\|u_n - Tu_n\|^2 + \|v_n - p_n\|^2\big) \end{aligned} \tag{3.8}$$

Similarly, we obtain

$$\|w_n' - y\|^2 \leq \|y_n - y\|^2 - \delta_n(2\gamma - \delta_n)\|A_{[n]}'y_n\|^2 - \beta(1-\beta)\big(\|u_n' - Su_n'\|^2 + \|v_n' - p_n'\|^2\big). \tag{3.9}$$

Substituting inequality (3.8) and inequality (3.9) in inequality (3.7) and using the assumptions on t_n and δ_n, we have that:

$$\begin{aligned} \|x_{n+1} - x\|^2 + \|y_{n+1} - y\|^2 &\leq (1-\alpha_n)(\|x_n - x\|^2 + \|y_n - y\|^2) + \alpha_n(\|u - x\|^2 + \|v - y\|^2) \\ &\quad - \delta_n(2\gamma - \delta_n)(1-\alpha_n)\big(\|A_{[n]}x_n\|^2 + \|A_{[n]}'y_n\|^2\big) \\ &\quad - \beta(1-\beta)(1-\alpha_n)\big(\|u_n' - Su_n'\|^2 + \|u_n - Tu_n\|^2\big) \\ &\quad - \beta(1-\beta)(1-\alpha_n)\big(\|v_n - p_n\|^2 + \|v_n' - p_n'\|^2\big) \\ &\quad - (1-\alpha_n)t_n\|Bw_n - B'w_n'\|^2 \\ &\quad - (1-\alpha_n)t_n(1-\eta-t_n)\big(\|d_n - Ud_n\|^2 + \|d_n' - Vd_n'\|^2\big) \\ &\leq (1-\alpha_n)(\|x_n - x\|^2 + \|y_n - y\|^2) \\ &\quad + \alpha_n(\|u - x\|^2 + \|v - y\|^2). \end{aligned} \tag{3.10}$$

Let $\Lambda_n(x,y) = \|x_n - x\|^2 + \|y_n - y\|^2$ and $K = \max\{\|u - x\|^2 + \|v - y\|^2, \|x_1 - x\|^2 + \|y_1 - y\|^2\}$. We now show by induction that $\Lambda_n(x,y) \leq K$ for all $n \geq 1$. Cleary, the statement is true for $n = 1$. Assume $\Lambda_n(x,y) \leq K$ up to some $n \geq 1$. Then, using inequality (3.10), we have that $\Lambda_{n+1}(x,y) \leq (1-\alpha_n)\Lambda_n(x,y) + \alpha_n(\|u - x\|^2 + \|v - y\|^2) \leq K$. Hence, $\{\Lambda_n(x,y)\}_{n=1}^{\infty}$ is bounded. Consequently, $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are bounded. Therefore, $\{(x_n, y_n)\}_{n=1}^{\infty}$ is bounded. □

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Theorem 3.2. *Let H_1 and H_2 be real Hilbert spaces and let X be a smooth real Banach space with dual space X^*. Let $U : H_1 \to H_1$ and $V : H_2 \to H_2$ be demimetric maps with constants η_1 and η_2 in $(-\infty, 1)$, respectively such that $(I - U)$ and $(I - V)$ are demi-closed at 0. Let $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be quasi-nonexpansive maps such that for each $(I - T)$ and $(I - S)$ are demi-closed at 0. Let $A_d : H_1 \to H_1,\ d = 1, 2, ..., p$ and $A'_h : H_2 \to H_2,\ h = 1, 2, ..., q$ be finite families of inverse strongly monotone maps with constants $\gamma_d > 0$ and $\gamma_h > 0$, respectively such that for each d and h, A_d and A'_h are demi-closed at 0. Let $\zeta_a : H_1 \to \mathbb{R}$ and $\zeta'_b : H_2 \to \mathbb{R}$ be finite families of convex and lower semi-continuous functions, $\nabla_a : H_1 \to H_1$ and $\nabla'_b : H_2 \to H_2$ be finite families of continuous monotone maps, $f_a : H_1 \times H_1 \to \mathbb{R}$ and $g_b : H_2 \times H_2 \to \mathbb{R}$ be finite families of bifunctions satisfying $(B1) - (B4)$, $\ a = 1, 2, ..., l,\ \ b = 1, 2, ..., m$. Let $B : H_1 \to X$ and $B' : H_2 \to X$ be bounded linear maps with adjoints B^* and B'^*, respectively. Let $\{(x_n, y_n)\}_{n=1}^{\infty}$ be a sequence in $H_1 \times H_2$ generated by Algorithm 1. Suppose Ω is nonempty and for some $\delta > 0$, t_n is such that*

$$\delta < t_n \leq \min\Big\{\rho, \frac{\|B'w'_n - Bw_n\|^2}{\|B^* J_X(Bw_n - B'w'_n)\|^2 + \|B'^* J_X(Bw_n - B'w'_n)\|^2}\Big\},\ \forall n \in \Gamma,$$

where $\Gamma := \{n \in \mathbb{N} : B'w'_n \neq Bw_n\}$; otherwise take $t_n = \min\{t, \rho\}$ (t is any positive number), where $\rho = \min\{1 - \eta, 2\gamma\}$, $\gamma = \min\{\gamma_1, \gamma_2\}$, $\gamma_1 = \min\{\gamma_d : d = 1, 2, ..., p\}$, $\gamma_2 = \min\{\gamma_h : h = 1, 2, ..., q\}$, $\eta = \max\{\eta_1, \eta_2\}$ and $\delta_n > 0$ is such that $\delta_n < 2\gamma$, then $\{(x_n, y_n)\}_{n=1}^{\infty}$ converges strongly to some point in Ω.

Proof. Let $(x^*, y^*) = P_\Omega(u, v)$. Using Lemma 2.6, we estimate as follows:

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n u + (1 - \alpha_n)e_n - x^*\|^2 \\
&\leq \|\alpha_n u + (1 - \alpha_n)e_n - x^* - \alpha_n(u - x^*)\|^2 + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle \\
&= \|(1 - \alpha_n)(e_n - x^*)\|^2 + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle \\
&\leq (1 - \alpha_n)\|e_n - x^*\|^2 + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle. \qquad (3.11)
\end{aligned}$$

Similarly, we obtain that:

$$\|y_{n+1} - y^*\|^2 \leq (1 - \alpha_n)\|e'_n - y^*\|^2 + 2\alpha_n\langle v - y^*, y_{n+1} - y^*\rangle. \qquad (3.12)$$

Adding inequality (3.11) and inequality (3.12), using inequalities (3.6), (3.8) and (3.9), we have that:

$$\begin{aligned}
\Lambda_{n+1}(x^*, y^*) &\leq (1 - \alpha_n)\Lambda_n(x^*, y^*) + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle + 2\alpha_n\langle v - y^*, y_{n+1} - y^*\rangle \\
&\quad -\delta_n(2\gamma - \delta_n)(1 - \alpha_n)\big(\|A_{[n]}x_n\|^2 + \|A'_{[n]}y_n\|^2\big) \\
&\quad -\beta(1 - \beta)(1 - \alpha_n)\big(\|u'_n - Su'_n\|^2 + \|u_n - Tu_n\|^2\big) \\
&\quad -\beta(1 - \beta)(1 - \alpha_n)\big(\|v_n - p_n\|^2 + \|v'_n - p'_n\|^2\big) - (1 - \alpha_n)t_n\|Bw_n - B'w'_n\|^2 \\
&\quad -(1 - \alpha_n)t_n(1 - \eta - t_n)\big(\|d_n - Ud_n\|^2 + \|d'_n - Vd'_n\|^2\big) \\
&\leq (1 - \alpha_n)\Lambda_n(x^*, y^*) + 2\alpha_n\langle x_n - x^*, u - x^*\rangle + 2\alpha_n\langle y_n - y^*, v - y^*\rangle \\
&\quad +2\alpha_n\|u - x^*\|\|x_{n+1} - x_n\| + 2\alpha_n\|v - y^*\|\|y_{n+1} - y_n\| \\
&\quad -\delta_n(2\gamma - \delta_n)(1 - \alpha_n)\big(\|A_{[n]}x_n\|^2 + \|A'_{[n]}y_n\|^2\big) \\
&\quad -\beta(1 - \beta)(1 - \alpha_n)\big(\|u'_n - Su'_n\|^2 + \|u_n - Tu_n\|^2\big) \\
&\quad -\beta(1 - \beta)(1 - \alpha_n)\big(\|v_n - p_n\|^2 + \|v'_n - p'_n\|^2\big) - (1 - \alpha_n)t_n\|Bw_n - B'w'_n\|^2 \\
&\quad -(1 - \alpha_n)t_n(1 - \eta - t_n)\big(\|d_n - Ud_n\|^2 + \|d'_n - Vd'_n\|^2\big) \qquad (3.13) \\
&\leq (1 - \alpha_n)\Lambda_n(x^*, y^*) + 2\alpha_n\big(\langle x_n - x^*, u - x^*\rangle + \langle y_n - y^*, v - y^*\rangle\big) \\
&\quad +2\alpha_n\big(\|u - x^*\|\|x_{n+1} - x_n\| + \|v - y^*\|\|y_{n+1} - y_n\|\big). \qquad (3.14)
\end{aligned}$$

We shall divide the rest of the proof into two cases.

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Case 1: Suppose there exists $N \in \mathbb{N}$ such that, $\forall\ n \geq N$, $\{\Lambda_n(x^*, y^*)\}_{n=1}^{\infty}$ is decreasing. Then, $\{\Lambda_n(x^*, y^*)\}_{n=1}^{\infty}$ is convergent. Utilizing the boundedness of $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$, the condition on t_n and the fact that $\lim\limits_{n\to\infty} \alpha_n = 0$, we obtain from inequality (3.13) that:

$$\lim_{n\to\infty} \|Bw_n - B'w_n'\| = 0, \quad \lim_{n\to\infty} \|A_{[n]}x_n\| = 0, \quad \lim_{n\to\infty} \|A'_{[n]}y_n\| = 0; \tag{3.15}$$

$$\lim_{n\to\infty} \|d_n - Ud_n\| = 0, \quad \lim_{n\to\infty} \|d_n' - Vd_n'\| = 0; \tag{3.16}$$

$$\lim_{n\to\infty} \|u_n' - Su_n'\| = 0, \quad \lim_{n\to\infty} \|u_n - Tu_n\| = 0, \tag{3.17}$$

$$\lim_{n\to\infty} \|v_n' - p_n'\| = 0, \quad \lim_{n\to\infty} \|v_n - p_n\| = 0. \tag{3.18}$$

Consequently, we have that:

$$\|d_n - e_n\| \leq t_n\|d_n - Ud_n\| \to 0, \quad \|d_n' - e_n'\| \leq t_n\|d_n' - Vd_n'\| \to 0, \quad \text{as} \quad n \to \infty.$$
$$\|d_n - w_n\| \leq t_n\|B^*\|\|Bw_n - B'w_n'\| \to 0, \quad \|d_n' - w_n'\| \leq t_n\|B'^*\|\|Bw_n - B'w_n'\| \to 0 \quad \text{as} \quad n \to \infty.$$
$$\|w_n - v_n\| = (1-\beta)\|v_n - p_n\| \to 0, \quad \|w_n' - v_n'\| = (1-\beta)\|v_n' - p_n'\| \to 0, \quad \text{as} \quad n \to \infty.$$
$$\|v_n - u_n\| = (1-\beta)\|u_n - Tu_n\| \to 0, \quad \|v_n' - u_n'\| = (1-\beta)\|u_n' - Su_n'\| \to 0, \quad \text{as} \quad n \to \infty.$$
$$\|u_n - x_n\| = \delta_n\|A_{[n]}x_n\| \to 0, \quad \|u_n' - y_n\| = \delta_n\|A'_{[n]}y_n\| \to 0, \quad \text{as} \quad n \to \infty.$$

Using boundedness of $\{e_n\}_{n=1}^{\infty}$ and the assumption that $\lim\limits_{n\to\infty} \alpha_n = 0$, we have that:

$$\|x_{n+1} - e_n\| = \alpha_n\|u - e_n\| \to 0, \qquad \|y_{n+1} - e_n'\| = \alpha_n\|v - e_n'\| \to 0 \ \text{ as } n \to \infty.$$

Then, we have that:

$$\begin{aligned}\|x_{n+1} - x_n\| \quad &\leq \quad \|x_{n+1} - e_n\| + \|e_n - d_n\| + \|d_n - w_n\| + \|w_n - v_n\| + \|v_n - u_n\| \\ &\quad + \|u_n - x_n\| \to 0, \quad \text{as} \quad n \to \infty.\end{aligned} \tag{3.19}$$

$$\|y_{n+1} - y_n\| \leq \|y_{n+1} - e_n'\| + \|e_n' - d_n'\| + \|d_n' - w_n'\| + \|w_n' - v_n'\| + \|v_n' - u_n'\| + \|u_n' - x_n'\| \to 0, \quad \text{as} \quad n \to \infty.$$

Claim: $\limsup_{n\to\infty} \left(\langle x_n - x^*, u - x^*\rangle + \langle y_n - y^*, v - y^*\rangle\right) \leq 0$.

Let $\{(x_{n_k}, y_{n_k})\}_{k=1}^{\infty}$ be a subsequence of $\{(x_n, y_n)\}_{n=1}^{\infty}$ such that

$$\begin{aligned}&\limsup_{n\to\infty} \left(\langle x_n - x^*, u - x^*\rangle + \langle y_n - y^*, v - y^*\rangle\right) \\ &= \lim_{k\to\infty} \left(\langle x_{n_k} - x^*, u - x^*\rangle + \langle y_{n_k} - y^*, v - y^*\rangle\right)\end{aligned}$$

Since $\{(x_{n_k}, y_{n_k})\}_{k=1}^{\infty}$ is bounded, then there exists a subsequence $\{(x_{n_{k_j}}, y_{n_{k_j}})\}_{j=1}^{\infty}$ of $\{(x_{n_k}, y_{n_k})\}_{k=1}^{\infty}$ that converges weakly to some point $(\hat{x}, \hat{y})$ in $H_1 \times H_2$, i.e., $x_{n_{k_j}} \rightharpoonup \hat{x}$ and $y_{n_{k_j}} \rightharpoonup \hat{y}$ as $j \to \infty$. Consequently, the subsequences: $\{u_{n_{k_j}}\}_{j=1}^{\infty}$ of $\{u_{n_k}\}_{k=1}^{\infty}$, $\{v_{n_{k_j}}\}_{j=1}^{\infty}$ of $\{v_{n_k}\}_{k=1}^{\infty}$, $\{p_{n_{k_j}}\}_{j=1}^{\infty}$ of $\{p_{n_k}\}_{k=1}^{\infty}$ $\{w_{n_{k_j}}\}_{j=1}^{\infty}$ of $\{w_{n_k}\}_{k=1}^{\infty}$, $\{e_{n_{k_j}}\}_{j=1}^{\infty}$ of $\{e_{n_k}\}_{k=1}^{\infty}$, and $\{d_{n_{k_j}}\}_{j=1}^{\infty}$ of $\{d_{n_k}\}_{k=1}^{\infty}$ converge weakly to $\hat{x}$. Also, the subsequences: $\{u'_{n_{k_j}}\}_{j=1}^{\infty}$ of $\{u'_{n_k}\}_{k=1}^{\infty}$, $\{v'_{n_{k_j}}\}_{j=1}^{\infty}$ of $\{v'_{n_k}\}_{k=1}^{\infty}$, $\{p'_{n_{k_j}}\}_{j=1}^{\infty}$ of $\{p'_{n_k}\}_{k=1}^{\infty}$ $\{w'_{n_{k_j}}\}_{j=1}^{\infty}$ of $\{w'_{n_k}\}_{k=1}^{\infty}$, $\{e'_{n_{k_j}}\}_{j=1}^{\infty}$ of $\{e'_{n_k}\}_{k=1}^{\infty}$, and $\{d'_{n_{k_j}}\}_{j=1}^{\infty}$ of $\{d'_{n_k}\}_{k=1}^{\infty}$ converge weakly to $\hat{y}$. Since $(I-T)$ and $(I-S)$ are demi-closed at 0, then using (3.17) we have that $(\hat{x}, \hat{y}) \in \Omega_1$.

Also, since $(I-U)$ and $(I-V)$ are demi-closed at 0, then using (3.16) we have that $\hat{x} \in F(U)$ and $\hat{y} \in F(V)$. By (3.15) and weak lower semi-continuity of $\|\|$, we have that $0 = \lim\limits_{j\to\infty} \|Bw_{n_{k_j}} - B'w'_{n_{k_j}}\| = \liminf_{j\to\infty} \|Bw_{n_{k_j}} - B'w'_{n_{k_j}}\| \geq \|B\hat{x} - B'\hat{y}\|$, which implies that $B\hat{x} = B'\hat{y}$. Therefore, $(\hat{x}, \hat{y}) \in \Omega_2$. Let $l \in \{1, 2, ..., p\}$. It is easy to see from (3.19) that

$$\lim_{j\to\infty} \|x_{n_{k_j+l}} - x_{n_{k_j}}\| = 0. \tag{3.20}$$

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Thus, using this, the fact that $A_l, l = 1,2,...,p$, is $\frac{1}{\gamma_l}-$Lipschitz continuous and (3.15), we have that

$$\begin{aligned} \|A_{[n_{k_j}]+l}x_{n_{k_j}}\| &\leq \|A_{[n_{k_j}]+l}x_{n_{k_j}} - A_{[n_{k_j}]+l}x_{n_{k_j}+l}\| + \|A_{[n_{k_j}]+l}x_{n_{k_j}+l}\| \\ &\leq \frac{1}{\gamma_1}\|x_{n_{k_j}} - x_{n_{k_j}+l}\| + \|A_{[n_{k_j}]+l}x_{n_{k_j}+l}\| \to 0 \text{ as } j \to \infty, \end{aligned} \tag{3.21}$$

Since $\{A_l\}_{l=1}^{p}$ is a finite family of maps and $A_{[n]} = A_{n \mod p}$, then for $l \in \{1,2,...,p\}$ there exists $z_l \in \{1,2,...,p\}$ such that $n_{k_j} + z_l = l \mod p$. Then using (3.21), we have that

$$\lim_{j\to\infty} \|A_l x_{n_{k_j}}\| = \lim_{j\to\infty} \|A_{[n_{k_j}]+l}x_{n_{k_j}}\| = 0, \quad l = 1,2,...,p. \tag{3.22}$$

Since for each $l = 1,2,...,p$, A_l is demi-closed at 0, using (3.22) we have that $A_l\hat{x} = 0$ for each $l = 1,2,...,p$. Similarly, we have that $A'_h\hat{y} = 0$ for each $h = 1,2,...,q$. Therefore, $(\hat{x},\hat{y}) \in \Omega_3$.

Next, we show that $(\hat{x},\hat{y}) \in \Omega_4$. Since

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = \lim_{n\to\infty} \|v_n - u_n\| = \lim_{n\to\infty} \|u_n - x_n\| = 0,$$

then one obtained that as $n \to \infty$,

$$\|v_{n+1} - v_n\| \leq \|v_{n+1} - u_{n+1}\| + \|u_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - u_n\| + \|u_n - v_n\| \to 0.$$

Consequently, for any $a \in \{1,2,...,l\}$, it is easy to see that

$$\lim_{j\to\infty} \|v_{n_{k_j}+a} - v_{n_{k_j}}\| = 0. \tag{3.23}$$

Using Lemma 2.10, we have that

$$\begin{aligned} \|v_{n_{k_j}} - G^{f_{[n_{k_j}]+a}}_{r_{[n_{k_j}]+a}}v_{n_{k_j}}\| &\leq \|v_{n_{k_j}} - v_{n_{k_j}+a}\| + \|v_{n_{k_j}+a} - G^{f_{[n_{k_j}]+a}}_{r_{[n_{k_j}]+a}}v_{n_{k_j}+a}\| \\ &\quad + \|G^{f_{[n_{k_j}]+a}}_{r_{[n_{k_j}]+a}}v_{n_{k_j}+a} - G^{f_{[n_{k_j}]+a}}_{r_{[n_{k_j}]+a}}v_{n_{k_j}}\| \\ &\leq 2\|v_{n_{k_j}} - v_{n_{k_j}+a}\| + \|v_{n_{k_j}+a} - G^{f_{[n_{k_j}]+a}}_{r_{[n_{k_j}]+a}}v_{n_{k_j}+a}\|. \end{aligned} \tag{3.24}$$

Using (3.23) and (3.18), one obtained from inequality (3.24) that

$$\lim_{j\to\infty} \|v_{n_{k_j}} - G^{f_{[n_{k_j}]+a}}_{r_{[n_{k_j}]+a}}v_{n_{k_j}}\| = 0. \tag{3.25}$$

Since $\{f_a\}_{a=1}^{l}$ is a finite family of maps and $f_{[n]} = f_{n \mod l}$, then for each $a \in \{1,2,...,l\}$ there exists $\omega_a \in \{1,2,...,l\}$ such that $n_{k_j} + \omega_a = n \mod l$, so that (3.25) becomes

$$\lim_{j\to\infty} \|v_{n_{k_j}} - G^{f_a}_{r_a}v_{n_{k_j}}\| = \lim_{j\to\infty} \|v_{n_{k_j}} - G^{f_{[n_{k_j}]+a}}_{r_{[n_{k_j}]+a}}v_{n_{k_j}}\| = 0, \quad a = 0,1,2,...,l. \tag{3.26}$$

Following similarly argument, we obtain that

$$\lim_{j\to\infty} \|v'_{n_{k_j}} - G^{g_b}_{r_b}v'_{n_{k_j}}\| = 0, \quad b = 1,2,...,m. \tag{3.27}$$

Since each $\{G^{f_a}_{r_a}\}_{a=1}^{l}$ and $\{G^{g_b}_{r_b}\}_{b=1}^{m}$ are finite families of nonexpansive maps, then using Lemma 2.4, (3.26), (3.27) and the fact that $\{(v_{n_{k_j}}, v'_{n_{k_j}})\}_{j=1}^{\infty}$ converges weakly to $(\hat{x},\hat{y})$, we conclude that $(\hat{x},\hat{y}) \in \Omega_4$. Therefore, $(\hat{x},\hat{y}) \in \Omega$. Consequently

$$\begin{aligned} &\limsup_{n\to\infty} \left(\langle x_n - x^*, u - x^*\rangle + \langle y_n - y^*, v - y^*\rangle\right) \\ &= \lim_{k\to\infty} \left(\langle x_{n_k} - x^*, u - x^*\rangle + \langle y_{n_k} - y^*, v - y^*\rangle\right) \\ &= \left(\langle \hat{x} - x^*, u - x^*\rangle + \langle \hat{y} - y^*, v - y^*\rangle\right) \\ &= \langle (\hat{x},\hat{y}) - (x^*,y^*), (u,v) - (x^*,y^*)\rangle \leq 0. \end{aligned}$$

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Hence, from inequality (3.14) and Lemma 2.8, we have that $\Lambda_n(x^*,y^*) \to 0$ as $n \to \infty$. Consequently, $\|x_n - x^*\| \to 0$ as $n \to \infty$ and $\|y_n - y^*\| \to 0$ as $n \to \infty$. Thus, $x_n \to x^*$ as $n \to \infty$ and $y_n \to y^*$ as $n \to \infty$.

Case 2: Suppose that there exists a subsequence $\{\Lambda_{n_j}(x^*,y^*)\}_{j=1}^{\infty}$ of $\{\Lambda_n(x^*,y^*)\}_{n=1}^{\infty}$ such that $\Lambda_{n_j}(x^*,y^*) < \Lambda_{n_{j+1}}(x^*,y^*)$ for all $j \geq 1$. Then, by Lemma 2.9, there exists a non-decreasing sequence $\{m_k\}_{k=1}^{\infty} \subset \mathbb{N}$ such that $m_k \to \infty$ as $k \to \infty$, $\Lambda_{m_k}(x^*,y^*) \leq \Lambda_{m_k+1}(x^*,y^*)$, and $\Lambda_k(x^*,y^*) \leq \Lambda_{m_k+1}(x^*,y^*)$ for all sufficiently large $k \in \mathbb{N}$. Since $\{x_{m_k}\}_{k=1}^{\infty}$ and $\{y_{m_k}\}_{k=1}^{\infty}$ are bounded, then following similar argument as in Case 1, we obtain that:

$$\limsup_{k\to\infty} \left(\langle x_{m_k} - x^*, u - x^*\rangle + \langle y_{m_k} - y^*, v - y^*\rangle\right) \leq 0.$$

Using the fact that $\Lambda_{m_k}(x^*,y^*) \leq \Lambda_{m_k+1}(x^*,y^*)$ for all $k \in \mathbb{N}$, we get from inequality (3.14) that

$$\begin{aligned} \alpha_{m_k}\Lambda_{m_k}(x^*,y^*) \quad &\leq \quad 2\alpha_{m_k}\left(\langle x_{m_k} - x^*, u - x^*\rangle + \langle y_{m_k} - y^*, v - y^*\rangle\right) \\ &\quad +2\alpha_{m_k}\left(\|x_{m_k+1} - x_{m_k}\|\|u - x^*\| + \|y_{m_k+1} - y_{m_k}\|\|v - y^*\|\right). \end{aligned}$$

Consequently, we get that $\Lambda_{m_k}(x^*,y^*) \to 0$ as $k \to \infty$. Since, $\Lambda_k(x^*,y^*) \leq \Lambda_{m_k+1}(x^*,y^*)$, then we have that $\Lambda_k(x^*,y^*) \to 0$ as $k \to \infty$. Therefore, $x_k \to x^*$ and $y_k \to y^*$ as $k \to \infty$. This completes the proof. □

For the next theorem, $\Omega_1 = \{(x,y) \in H_1 \times H_2 : (x,y) \in \left(\cap_{i=1}^{\infty}(T_i)\right) \times \left(\cap_{i=1}^{\infty}(S_i)\right)\}$.

Theorem 3.3. *Let H_1 and H_2 be real Hilbert spaces and let X be a smooth real Banach space with dual space X^*. Let $U : H_1 \to H_1$ and $V : H_2 \to H_2$ be demimetric maps with constants η_1 and η_2 in $(-\infty, 1)$, respectively such that $(I - U)$ and $(I - V)$ are demi-closed at 0. Let $T_t : H_1 \to H_1$, $t = 1, 2, \ldots$ and $S_z : H_2 \to H_2$, $z = 1, 2, \ldots$ be infinite families of quasi-nonexpansive maps such that for each t and z $(I - T_t)$ and $(I - S_z)$ are demi-closed at 0. Let $A_d : H_1 \to H_1$, $d = 1, 2, \ldots, p$ and $A'_h : H_2 \to H_2$, $h = 1, 2, \ldots, q$ be finite families of inverse strongly monotone maps with constants $\gamma_d > 0$ and $\gamma_h > 0$, respectively such that for each d and h, A_d and A'_h are demi-closed at 0. Let $\zeta_a : H_1 \to \mathbb{R}$ and $\zeta'_b : H_2 \to \mathbb{R}$ be finite families of convex and lower semi-continuous functions, $\nabla_a : H_1 \to H_1$ and $\nabla'_b : H_2 \to H_2$ be finite families of continuous monotone maps, $f_a : H_1 \times H_1 \to \mathbb{R}$ and $g_b : H_2 \times H_2 \to \mathbb{R}$ be finite families of bifunctions satisfying $(B1) - (B4)$, $a = 1, 2, \ldots, l$, $b = 1, 2, \ldots, m$. Let $B : H_1 \to X$ and $B' : H_2 \to X$ be bounded linear maps with adjoints B^* and B'^*, respectively. Let $\{(x_n, y_n)\}_{n=1}^{\infty}$ be a sequence in $H_1 \times H_2$ generated by Algorithm 1. Suppose Ω is nonempty and t_n is such that for $\delta > 0$ (small enough),*

$$\delta < t_n \leq \min\left\{\rho, \frac{\|B'w'_n - Bw_n\|^2}{\|B^*J_X(Bw_n - B'w'_n)\|^2 + \|B'^*J_X(Bw_n - B'w'_n)\|^2}\right\}, \ \forall n \in \Gamma,$$

where $\Gamma := \{n \in \mathbb{N} : B'w'_n \neq Bw_n\}$, otherwise take $t_n = \min\{t, \rho\}$ (t is any positive number), where $\rho = \min\{1 - \eta, 2\gamma\}$, $\gamma = \min\{\gamma_1, \gamma_2\}$, $\gamma_1 = \min\{\gamma_d : d = 1, 2, \ldots, p\}$, $\gamma_2 = \min\{\gamma_h : h = 1, 2, \ldots, q\}$, $\eta = \max\{\eta_1, \eta_2\}$ and $\delta_n > 0$ is such that $\delta_n < 2\gamma$, then $\{(x_n, y_n)\}_{n=1}^{\infty}$ converges strongly to some point in Ω.

Proof. Define $Tx = \sum_{i=1}^{\infty}\zeta_i T_i x$ and $Sx = \sum_{i=1}^{\infty}\zeta_i S_i x$ where $\{\zeta_i\}_{i=1}^{\infty}$ is a sequence in $(0,1)$ such that $\sum_{i=1}^{\infty}\zeta_i = 1$. Then, by Lemma 2.11, T and S are quasi-nonexpansive, $(I - T)$ and $(I - S)$ are demi-closed at zero, $F(T) = \cap_{i=1}^{\infty}F(T_i)$ and $F(S) = \cap_{i=1}^{\infty}F(S_i)$. Hence, the conclusion follows from Theorem 3.1 and Theorem 3.2. □

4 Discussion and conclusion

Although the result of Ofoedu and Araka [9] is a unification and generalization of a host of important recent results, however, it is worth noting that in order to implement their algorithm, one needs to compute first the norms of A and A', which in general, is a much more difficult task than solving the original problem. We have constructed an iterative algorithm with a step-size independent of the norm of the operators that approximates a common point in: the set of solutions of SEFPP involving η-demimetric maps, the set of common zeros of finite families of inverse strongly monotone maps, the set of common solutions of systems of generalized mixed equilibrium problems, and the set of common fixed points of infinite families of quasi-nonexpansive maps.The diversity of life calls for different ways of Problem solving and our Algorithms and Theorems have provided for feasibility.

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6 Competing Interests

The authors declare that they have no competing interests.

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