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On the Monoid Generated by Newton Algorithm in Solving a Particular Cubic Polynomial

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Abstract

We initiate the study of the algebraic interpretations of the Newton algorithm via transformation semigroup. We use MATLAB to solve the equation $x^3 + 4x^2 - 10 = 0$ with an error tolerance of $\epsilon = 10^{-4}$. In each iteration, we obtain a transformation on a set of seven elements. With the aid of GAP 4.0, we construct a monoid that interprets the algebraic phenomena of all the iterations. The study reveals that the monoid obtained is left-adequate with 64 elements.

Keywords: Rank properties, Order decreasing and Monotone transformations, Abundant semigroup.
MSC2010: 20M20.

1 Introduction

Transformation semigroups are mathematical constructs that emerge in the analysis of dynamical systems, particularly in relation to operations on sets. A *semigroup* is defined as a set that is paired with an associative binary operation. When a set consists of transformations of a given set and is closed together with composition of functions, such a set is referred to as *a transformation semigroup*. For chains, which are characterized as totally ordered sets, transformation semigroups can be distinguished by their action on these ordered structures. They aid in investigating how transformations influence the order and arrangement of chains. For instance, a transformation semigroup can represent processes like renaming or rearranging elements while maintaining specific order properties. The study of these semigroups includes exploring the characteristics of the transformations, their compositions, and any fixed points or invariant elements present. For a comprehensive introduction to the basic ideas of semigroup theory, we recommend that the reader consult the textbooks of Howie [1] and Higgins [2].

Newton's algorithm, commonly known as the *Newton-Raphson method*, is an iterative numerical strategy employed to approximate the roots of real-valued functions, particularly nonlinear equations. Given a function $f(x)$ and its derivative $f'(x)$, the algorithm begins with an initial estimate x_0 close to the desired root. The subsequent approximation x_{n+1} is computed using the formula:

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$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

This procedure is repeated until the approximations converge to a sufficiently precise root. The method is notably effective due to its quadratic convergence near the root, which indicates that the number of accurately computed digits roughly doubles at each iteration, provided the initial guess is adequately close and the derivative does not approach zero. However, the method may fail if the initial estimate is misguided or if $f'(x_n)$ equals zero.

A *transformation* of a given set say X is a map from X into X. A transformation is *partial* if its domain is a subset of X; and it is *total* (or *full*) if its domain is the whole of X. It is a known fact that the collection of all partial transformations and the collection of all full transformations on X are semigroups under the usual function composition. Properties such as algebraic and combinatorial, as well as rank properties of various transformation semigroups have been investigated by various authors, for example, see [3–9].

One of the questions frequently raised among pure and applied mathematicians concerns the interrelationship between mathematical concepts and ideas. It is well known that applied and pure mathematics support each other, necessitating a collaborative approach. I am pleased to initiate a study of specific algorithms to uncover their tendencies to exhibit certain algebraic properties. The goal is to categorize these algorithms into various algebraic structures. We firmly believe that this is achievable, and in doing so, this concept may reveal certain algebraic structures that are either hidden or novel to us.

2 Preliminaries

We are going to consider a nonlinear equation given by

$$x^3 + 4x^2 - 10 = 0, \tag{2.1}$$

which was solved using the Newton method, known as the *Newton-Raphson method*, named after Isaac Newton and Joseph Raphson, with error tolerance $\epsilon = 10^{-4}$ and initial guess of 0.3. This root-finding algorithm generates successive approximations to the root of a real-valued function. Equation (2.1) was solved both manually and using MATLAB version 2018, yielding approximate results, see **Appendix B**. We define the set $\{x_0, x_1, \ldots, x_6\}$ as the base chain for the approximations, consisting of 7 elements. Each step of the algorithm is described as follows:

- **Step 1:** The initial point x_0 is given, interpreted as x_0 mapping to itself, denoted $x_0 \to x_0$, while all other points remain fixed.
- **Step 2:** $x_0 \to x_1$, with all other points remaining fixed;
- **Step 3:** $x_0 \to x_1$, $x_1 \to x_2$, while all other points remain fixed;
- **Step 4:** $x_0 \to x_1$, $x_1 \to x_2$, $x_2 \to x_3$, with all other points remaining fixed;
- **Step 5:** $x_0 \to x_1$, $x_1 \to x_2$, $x_2 \to x_3$, $x_3 \to x_4$, while all other points remain fixed;
- **Step 6:** $x_0 \to x_1$, $x_1 \to x_2$, $x_2 \to x_3$, $x_3 \to x_4$, $x_4 \to x_5$, with all other points remaining fixed;
- **Step 7:** $x_0 \to x_1$, $x_1 \to x_2$, $x_2 \to x_3$, $x_3 \to x_4$, $x_4 \to x_5$, $x_5 \to x_6$.

For simplicity, we denote the set $\{x_0, x_1, \ldots, x_6\}$ by the chain $\{1, \ldots, 7\}$. We observed that the behavior of the algorithm with respect to this problem establishes an order of the form $1 \leq 2 \leq \cdots \leq 7$, where each element represents the STEP of the iteration in the solution of equation (2.1). Considering that the algorithm terminates at a certain tolerance level, we obtained the following

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transformations in each iteration from the algorithm. The transformations are given in two line notation as follows:

Step 1: Produces the transformation: $\alpha_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$;

Step 2: Produces the transformation: $\alpha_2 = \begin{pmatrix} \{1,2\} & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$;

Step 3: Produces the transformation: $\alpha_3 = \begin{pmatrix} 1 & \{2,3\} & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$;

Step 4: Produces the transformation: $\alpha_4 = \begin{pmatrix} 1 & 2 & \{3,4\} & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$;

Step 5: Produces the transformation: $\alpha_5 = \begin{pmatrix} 1 & 2 & 3 & \{4,5\} & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$;

Step 6: Produces the transformation: $\alpha_6 = \begin{pmatrix} 1 & 2 & 3 & 4 & \{5,6\} & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$; and finally,

Step 7: Produces the transformation: $\alpha_7 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \{67\} \\ 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$.

Let $N = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$. Here, N represents the set of transformations generated by the algorithm. We then state the following lemma:

Lemma 2.1. *N is not a semigroup.*

Proof. Consider $\alpha_3, \alpha_6 \in N$; then their product is given by:

$$\alpha_3 \cdot \alpha_6 = \begin{pmatrix} 1 & \{2,3\} & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & \{5,6\} & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & \{5,6\} & 7 \\ 3 & 4 & 4 & 5 & 6 & 7 \end{pmatrix} \notin N.$$

□

Having established that N is not a semigroup, it becomes important to explore the semigroup generated by N. This investigation will provide insights into the type of semigroup produced by the Newton algorithm and reveal the algebraic behavior of this algorithm.

Additionally, it would be intriguing to encounter a semigroup with unknown algebraic properties. Describing its characteristics would undoubtedly enhance our understanding of algebra.

Notice that if S is a semigroup and $T \subseteq S$ then $\langle T \rangle$ (reads *semigroup generated by* T) is a subsemigroup of S, and that every subset of a semigroup generate a semigroup see [10]. In fact the generating set of various semigroups of transformations on a finite chain $[n]$ have been investigated by various authors, for example see [11, 12]. Using GAP 4.0, we discovered that $\langle N \rangle$ contains 64 elements (refer to Appendix A). We have listed these elements and obtained the following remark.

Remark 2.2. Every element in $\langle N \rangle$ is a full contraction *(*a transformation is called *a full contraction* if (for all $x, y \in [n]$) $|x\alpha - y\alpha| \leq |x - y|$*)* and hence every element in $\langle N \rangle$ is a contraction. Consequently $\langle N \rangle$ is a subsemigroup of the full contraction semigroup $\mathcal{CT}_7$*, see [13].*

For brevity, we shall write $\mathcal{N} = \langle N \rangle$. An element a in a semigroup S is said to be *regular*, if there is $b \in S$ such that $a = bab$; and if every element of S is regular, then S is called a *regular semigroup*. For a transformation $\alpha \in \mathcal{N}$, we shall denote $\ker \alpha = \{(x, y) \in [n] \times [n] : x\alpha = y\alpha\}$, $\operatorname{Im} \alpha$ to denote *the image set* of α, and $E(S)$ will denote the collection of idempotents in S. Each time a new class of semigroup is encountered, the first algebraic question that generally arises concerns its regularity. We will explore this question in the next remark.

Remark 2.3. *The element $\alpha_2 \in \mathcal{N}$ is not regular. Indeed observe that α_2 is* regular *if and only if $\alpha_2 = \alpha_2 \beta \alpha_2$.*

Thus, β must be one of the transformations $\begin{pmatrix} \{1,2\} & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}, \begin{pmatrix} \{1,3\} & 2 & 4 & 5 & 6 & 7 \\ 3 & 1 & 4 & 5 & 6 & 7 \end{pmatrix}, \begin{pmatrix} \{1,4\} & 2 & 3 & 5 & 6 & 7 \\ 4 & 1 & 3 & 5 & 6 & 7 \end{pmatrix}, \begin{pmatrix} \{1,5\} & 2 & 3 & 4 & 6 & 7 \\ 5 & 1 & 3 & 4 & 6 & 7 \end{pmatrix}, \begin{pmatrix} \{1,6\} & 2 & 3 & 4 & 5 & 7 \\ 6 & 1 & 3 & 4 & 5 & 7 \end{pmatrix}, \begin{pmatrix} \{1,7\} & 2 & 3 & 4 & 5 & 6 \\ 7 & 1 & 3 & 4 & 5 & 6 \end{pmatrix}$.

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However, none of the maps $\begin{pmatrix} \{1,2\} & 3 & 4 & 5 & 6 & 7 \\ 1 & & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$, $\begin{pmatrix} \{1,3\} & 2 & 4 & 5 & 6 & 7 \\ 3 & 1 & 4 & 5 & 6 & 7 \end{pmatrix}$, $\begin{pmatrix} \{1,4\} & 2 & 3 & 5 & 6 & 7 \\ 4 & 1 & 3 & 5 & 6 & 7 \end{pmatrix}$, $\begin{pmatrix} \{1,5\} & 2 & 3 & 4 & 6 & 7 \\ 5 & 1 & 3 & 4 & 6 & 7 \end{pmatrix}$, $\begin{pmatrix} \{1,6\} & 2 & 3 & 4 & 5 & 7 \\ 6 & 1 & 3 & 4 & 5 & 7 \end{pmatrix}$, $\begin{pmatrix} \{1,7\} & 2 & 3 & 4 & 5 & 6 \\ 7 & 1 & 3 & 4 & 5 & 6 \end{pmatrix}$ *is in* $\mathcal{N}$.

An element $a \in S$ is an *idempotent* if $a^2 = a$. Now that we have established the monoid is not regular, we provide the necessary and sufficient condition for an element to be regular in the following theorem:

Theorem 2.4. *An element in the monoid* $\mathcal{N}$ *is regular if and only if it is an idempotent.*

3 Main Results

3.1 Green's and Starred Green's relation

Our primary interest lies in examining the algebraic behavior of the transformations generated by Newton's algorithm when solving equation (2.1). We are motivated to explore this because Green's relations provide an effective method for categorizing elements based on their algebraic properties. These relations were introduced by J. A. Green to sort out elements of a semigroup. They have proven effective in categorizing elements in every semigroup containing idempotents, as they are defined in terms of the principal ideals (left, right, or two-sided ideals) generated by elements within a semigroup. For the basic definition of these relations or any undefined terms, we refer readers to Howie [1].

As an easy exercise, one can see from the elements of $\mathcal{N}$ that the monoid $\mathcal{N}$ is $\mathcal{J}-$trivial, meaning that all its elements are along the diagonal of the egg-box provided by the Green's relations. Thus, the lemma below follows.

Lemma 3.1. *The monoid* $\mathcal{N}$ *is* $\mathcal{J}-$*trivial.*

Proof. Let $(\alpha, \beta) \in \mathcal{L}$, then it is routine matter to show that $\text{Im}\ \alpha = \text{Im}\ \beta$. Clearly (see Appendix A) $\ker \alpha = \ker \beta$ from elements of same height (i.e., height means the number of elements in the image set) which ensures that $\alpha = \beta$.

Conversely, if $\alpha = \beta$, then since $\mathcal{N}$ is a monoid, it follows that $\alpha = Id_{[7]}\beta$ and $\beta = Id_{[7]}\alpha$, where $Id_{[7]} = \alpha_7$ is the identity element. Thus, $(\alpha, \beta) \in \mathcal{L}$, as required.

We can in a similar way show that $\mathcal{N}$ is $\mathcal{R}-$trivial and $\mathcal{H}-$trivial, and so it is $\mathcal{J}-$trivial. □

Now in order to investigate the algebraic class to which to monoid $\mathcal{N}$ belongs, we need to further characterize the starred Green's equivalences. The five Green's equivalences are: $\mathcal{L}^*$, $\mathcal{R}^*$, $\mathcal{D}^*$, $\mathcal{J}^*$, and $\mathcal{H}^*$. The relation $\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*$, while $\mathcal{D} = \mathcal{L}^* \circ \mathcal{R}^*$. In finite non-regular semigroups, the composition may not commute. For the fundamental properties of these relations, we will recommend for the reader Fountain [14]. There are many non-regular transformation semigroups whose starred Green's relations have been characterized and are used to obtain their rank properties, for example see Zubairu *et. al.,* [13].

We now present the following lemma.

Lemma 3.2. *For* $\alpha, \beta \in \mathcal{N}$ *we have:*

(i) $\alpha \mathcal{L}^* \beta$ *if and only* $\text{Im}\ \alpha = \text{Im}\ \beta$*;*

(ii) $\alpha \mathcal{R}^* \beta$ *if and only if* $\ker \alpha = \ker \beta$*;*

(iii) $\alpha \mathcal{H}^* \beta$ *if and only if* $\alpha = \beta$*;*

(iv) $\alpha \mathcal{D}^* \beta$ *if and only if* $|\text{Im}\ \alpha| = |\text{Im}\ \beta|$.

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Proof. The proof is straightforward from the definitions of these relations and characterizations of the Green's relations from Lemma 3.1. □

The relation $\mathcal{D}^*$ have the following characterization on N.

Theorem 3.3. *On the monoid $\mathcal{N}$, $\mathcal{D}^* = \mathcal{L}^*$.*

Theorem 3.4. *On the monoid $\mathcal{N}$, $(\mathcal{L}^* \circ \mathcal{R}^*)^2 = \mathcal{L}^* \circ \mathcal{R}^*$.*

Proof. To show this, it is enough to show that $\mathcal{L}^* \circ \mathcal{R}^* = \mathcal{R}^* \circ \mathcal{L}^*$. Suppose $(\alpha, \beta) \in \mathcal{L}^* \circ \mathcal{R}^*$. It means that there exists $\gamma \in \mathcal{N}$ such that $\alpha \mathcal{L}^* \gamma \mathcal{R}^* \beta$, which implies $\alpha \mathcal{L}^* \gamma$ and $\gamma \mathcal{R}^* \beta$. Now, $\gamma \mathcal{R}^* \beta$ means that $\gamma = \beta$ (From Appendix A, since no two different elements having the same kernel class). Thus, $\beta \mathcal{R}^* \beta$. Notice that $\alpha \mathcal{L}^* \beta$, and so $\alpha \mathcal{R}^* \alpha \mathcal{L}^* \beta$ i.e., $(\alpha, \beta) \in \mathcal{R}^* \circ \mathcal{L}^*$. Therefore, $\mathcal{L}^* \circ \mathcal{R}^* \subseteq \mathcal{R}^* \circ \mathcal{L}^*$.

On the other hand suppose $(\alpha, \beta) \in \mathcal{R}^* \circ \mathcal{L}^*$. This means that there exists $\gamma\prime \in \mathcal{N}$ such that $\alpha \mathcal{R}^* \gamma\prime \mathcal{L}^* \beta$. i.e., $\alpha \mathcal{R}^* \gamma\prime$ and $\gamma\prime \mathcal{L}^* \beta$. Notice that $\alpha \mathcal{R}^* \gamma\prime$ means $\alpha = \gamma\prime$. However, $\alpha \mathcal{L}^* \beta$ and by Lemma 3.1 $\beta \mathcal{R}^* \beta$. i.e., $\alpha \mathcal{L}^* \beta$ and $\beta \mathcal{R}^* \beta$. i.e., $(\alpha, \beta) \in \mathcal{L}^* \circ \mathcal{R}^*$. Therefore, $\mathcal{R}^* \circ \mathcal{L}^* \subseteq \mathcal{L}^* \circ \mathcal{R}^*$. Hence, $\mathcal{R}^* \circ \mathcal{L}^* = \mathcal{L}^* \circ \mathcal{R}^*$, as required. □

A semigroup S is said to be *left abundant* if each $\mathcal{L}^*$-class contains an idempotent; it is said to be *right abundant* if each $\mathcal{R}^*$-class contains an idempotent; and it is *abundant* if it is both left and right abundant. These classes of semigroups were introduced by Fountain [10, 14]. On the monoid $\mathcal{N}$ we have actually obtain the following result using Lemma 3.2.

Lemma 3.5. *The monoid $\mathcal{N}$ is left abundant.*

Proof. It is easy to observe from Appendix **A** that every $\mathcal{L}^*-$ class contain idempotent. □

We have the following remark.

Remark 3.6. *It is important to note that the monoid $\mathcal{N}$ is not right abundant, this is due to the fact that, if we consider the element*

$$\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 4 & 5 & 6 & 7 \end{pmatrix} \in \mathcal{N}.$$

Then

$$R_\delta^* = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 4 & 5 & 6 & 7 \end{pmatrix} \right\}$$

has no idempotent element.

The idempotents elements in the monoid $\mathcal{N}$ are as follows. (and are denoted as):

$$e_1 = \begin{pmatrix} \{1,2\} & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}, \quad e_2 = \begin{pmatrix} \{1,2,3\} & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 7 \end{pmatrix},$$

$$e_3 = \begin{pmatrix} \{1,2,3,4\} & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 \end{pmatrix}, \quad e_4 = \begin{pmatrix} \{1,2,3,4,5\} & 6 & 7 \\ 5 & 6 & 7 \end{pmatrix},$$

$$e_5 = \begin{pmatrix} \{1,2,3,4,5,6\} & 7 \\ 6 & 7 \end{pmatrix}, \quad e_6 = \begin{pmatrix} \{1,2,3,4,5,6,7\} \\ 7 \end{pmatrix}.$$

Idempotents are pivotal in understanding the algebraic structure of any given semigroup that contains them. The significance of studying these elements dates back to works like those found in [15, 16]. In our case, we denote the set of idempotents within $\mathcal{N}$ as $E(\mathcal{N}) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$. An abundant semigroup whose set of idempotents forms a commutative structure known as a *semilattice* (where every element is an idempotent) is often termed an adequate semigroup. Our goal here involves determining whether $E(\mathcal{N})$ constitutes such a structure through examination via its product table. Should it do so, this discovery would shed light on additional algebraic characteristics inherent within our monoid under investigation.

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Moreover, exploring how these relationships might generalize could potentially unveil yet more intriguing properties worthy of consideration. Now let's construct and analyze this multiplication table:

3.2 The Cayley table shows composition of elements in $E(\mathcal{N})$

$\circ$	e_1	e_2	e_3	e_4	e_5	e_6
e_1	e_1	e_2	e_3	e_4	e_5	e_6
e_2	e_2	e_2	e_3	e_4	e_5	e_6
e_3	e_3	e_3	e_3	e_4	e_5	e_6
e_4	e_4	e_4	e_4	e_4	e_5	e_6
e_5	e_5	e_5	e_5	e_5	e_5	e_6
e_6	e_6	e_6	e_6	e_6	e_6	e_6

Table 1: Composition of elements in $E(\mathcal{N})$

It is now very clear from the above Table 1 that the following result is true.

Lemma 3.7. *$E(\mathcal{N})$ is a semilattice.*

Consequently, we have proved the following result.

Theorem 3.8. *The monoid $\mathcal{N}$ is left adequate.*

Proof. The result follows from Lemma 3.5 and Lemma 3.7. □

4 Conclusion

It is now clear to us that the semigroup generated by Newton's algorithm in solving equation (2.1) is a left abundant semigroup. As such, we can phenomenologically say that one can decode the algebraic properties of a given algorithm. One can now raise the following questions. Given any algorithm, can we decode its resultant algebraic property?

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A Appendix A

Let $\alpha_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$.

Then $\mathcal{L}^* - class$ of α_1 is:

$$\mathcal{L}^*_{\alpha_1} = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 4 & 5 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 5 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 7 \end{pmatrix} \right\}$$

Consider $\alpha_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 3 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$. Then the $\mathcal{L}^* - class$ of α_2 denoted by $\mathcal{L}^*_{\alpha_2}$ is:

$$\mathcal{L}^*_{\alpha_2} = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 3 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 3 & 4 & 4 & 5 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 3 & 4 & 5 & 5 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 3 & 4 & 5 & 6 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 3 & 4 & 5 & 6 & 7 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 4 & 4 & 5 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 4 & 5 & 5 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 4 & 5 & 6 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 4 & 5 & 6 & 7 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 5 & 5 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 5 & 6 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 5 & 6 & 7 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 6 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 6 & 7 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 7 & 7 & 7 \end{pmatrix} \right\}.$$

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Now consider $\alpha_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 4 & 4 & 4 & 5 & 6 & 7 \end{pmatrix}$. Then

$$\mathcal{L}^*_{\alpha_3} = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 4 & 4 & 4 & 5 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 4 & 4 & 5 & 5 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 4 & 4 & 5 & 6 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 4 & 4 & 5 & 6 & 7 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 4 & 5 & 5 & 5 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 4 & 5 & 5 & 6 & 7 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 4 & 5 & 6 & 6 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 4 & 5 & 5 & 6 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 4 & 5 & 6 & 6 & 7 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 4 & 5 & 6 & 7 & 7 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 5 & 5 & 5 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 5 & 5 & 6 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 5 & 5 & 6 & 7 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 5 & 6 & 6 & 7 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 5 & 6 & 6 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 5 & 6 & 7 & 7 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 6 & 6 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 6 & 7 & 7 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 7 & 7 & 7 \end{pmatrix} \right\}.$$

Let $\alpha_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 5 & 5 & 5 & 5 & 6 & 7 \end{pmatrix}$. Then

$$\mathcal{L}^*_{\alpha_4} = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 5 & 5 & 5 & 5 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 5 & 5 & 5 & 6 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 5 & 5 & 5 & 6 & 7 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 5 & 5 & 6 & 6 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 5 & 5 & 6 & 6 & 7 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 5 & 5 & 6 & 7 & 7 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 5 & 6 & 6 & 7 & 7 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 5 & 6 & 6 & 6 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 5 & 6 & 6 & 6 & 7 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 5 & 6 & 7 & 7 & 7 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 6 & 6 & 6 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 6 & 6 & 6 & 7 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 6 & 7 & 7 & 7 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 7 & 7 & 7 & 7 \end{pmatrix} \right\}.$$

If we consider $\alpha_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 6 & 6 & 6 & 6 & 6 & 7 \end{pmatrix}$ then

$$\mathcal{L}^*_{\alpha_5} = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 6 & 6 & 6 & 6 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 6 & 6 & 6 & 6 & 7 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 6 & 6 & 6 & 7 & 7 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 6 & 6 & 7 & 7 & 7 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 6 & 7 & 7 & 7 & 7 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 7 & 7 & 7 & 7 & 7 \end{pmatrix} \right\}.$$

For the element $\alpha_6 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 7 & 7 & 7 & 7 & 7 & 7 \end{pmatrix}$, $L^*_{\alpha_6} = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 7 & 7 & 7 & 7 & 7 & 7 \end{pmatrix} \right\}$.

Lastly, the identity element $\alpha_7 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$ has $L^*_{\alpha_7} = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix} \right\}$.

B Appendix B

Function: `newton_cubic`

```
function newton_cubic()
```

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```
    % Define function and derivative
    f  = @(x) x.^3 + 4*x.^2 - 10;
    df = @(x) 3*x.^2 + 8*x;

    % Settings
    tol = 1e-4;
    maxit = 100;

    % Two extreme initial guesses: a real start and a complex start
    initials = [0.3,-2+5i ];

    for k = 1:numel(initials)
        x0 = initials(k);
        fprintf('\nRun %d: initial guess x0 = %s\n', k, num2str(x0));
        [root, iterates] = newton_stepwise(f, df, x0, tol, maxit);
        % Print final result
        fprintf('Converged root = %s (after %d iterations)\n', num2str(root), numel(iterates)-1);
    end
end
```

Function: `newton_stepwise`

```
function [x, iterates] = newton_stepwise(f, df, x0, tol, maxit)
    x = x0;
    iterates = x;
    fprintf(' iter   x_k\n');
    fprintf('%4d   %s\n', 0, num2str(x));
    for it = 1:maxit
        fx = f(x);
        dfx = df(x);
        if dfx == 0
            error('Zero derivative encountered at iteration %d (x = %s).', it, num2str(x));
        end
        xnew = x - fx./dfx;
        iterates(end+1) = xnew; %#ok<AGROW>
        fprintf('%4d   %s   f(x)=%s\n', it, num2str(xnew), num2str(f(xnew)));
        if abs(xnew - x) < tol
            x = xnew;
            return
        end
        x = xnew;
    end
    warning('Maximum iterations reached without full convergence.');
end
```

Ith tolerance: 10^{-4}
Initial point: 0.3
Run : Initial guess $x_0 = 0.3$
Converged root: 1.3652 (after 7 iterations)

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iter	x_k	f(x)
0	0.3	-
1	3.9004	110.1878
2	2.4664	29.3366
3	1.6940	6.34
4	1.4079	0.71992
5	1.3661	0.014317
6	1.3652	6.0762×10^{-6}

Table 2: Newton's Method Iterations

C GAP Codes

```
gap> LoadPackage("semigroups");

true
gap>  S := Semigroup(Transformation([1, 2, 3, 3]),
>                    rec(hashlen := 100003, small := false));
<commutative transformation semigroup of degree 4 with 1 generator>
gap> S := Semigroup(Transformation([2, 1]));
<commutative transformation semigroup of degree 2 with 1 generator>
gap> s;
Error, Variable: 's' must have a value
not in any function at *stdin*:5
gap> S;
<commutative transformation semigroup of degree 2 with 1 generator>
gap> S := Semigroup(Transformation([2, 1]
> Transformation([1,2]);
Syntax error: ) expected
Transformation([1,2]);
^^^^^^^^^^^^^^
gap> S := Semigroup(Transformation([1, 2, 3, 4, 5, 6, 7]),
> Transformation([2, 2, 3, 4, 5, 6, 7]), Transformation([2, 3, 3, 4, 5, 6, 7]),
> Transformation([2, 3, 4, 4, 5, 6, 7]), Transformation([2, 3, 4, 5, 5, 6, 7]),
> Transformation([2, 3, 4, 5, 6, 6, 7]), Transformation([2, 3, 4, 5, 6, 7, 7]));
<transformation monoid of degree 7 with 6 generators>
gap> Size(S);
64
gap> GreensDClasses(S);
[ <Green's D-class: IdentityTransformation>, <Green's D-class: Transformation( [ 2, 2 ] )>,
<Green's D-class: Transformation( [ 2, 3, 3 ] )>
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      <Green's L-class: Transformation( [ 4, 5, 5, 5, 6, 6 ] )>,
      <Green's L-class: Transformation( [ 4, 5, 5, 5, 6, 7, 7 ] )>,
       <Green's L-class: Transformation( [ 4, 5, 5, 6, 6, 6 ] )>,
      <Green's L-class: Transformation( [ 4, 5, 5, 6, 6, 7, 7 ] )>,
       <Green's L-class: Transformation( [ 4, 5, 5, 6, 7, 7, 7 ] )>,
      <Green's L-class: Transformation( [ 4, 5, 6, 6, 6, 6 ] )>,
      <Green's L-class: Transformation( [ 4, 5, 6, 6, 6, 7, 7 ] )>,
      <Green's L-class: Transformation( [ 4, 5, 6, 6, 7, 7, 7 ] )>,
      <Green's L-class: Transformation( [ 4, 5, 6, 7, 7, 7, 7 ] )>,
      <Green's L-class: Transformation( [ 5, 5, 5, 5, 5 ] )>,
      <Green's L-class: Transformation( [ 5, 5, 5, 5, 6, 6 ] )>,
       <Green's L-class: Transformation( [ 5, 5, 5, 5, 6, 7, 7 ] )>,
```

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```
      <Green's L-class: Transformation( [ 5, 5, 5, 6, 6, 6 ] )>,
       <Green's L-class: Transformation( [ 5, 5, 5, 6, 6, 7, 7 ] )>,
       <Green's L-class: Transformation( [ 5, 5, 5, 6, 7, 7, 7 ] )>,
        <Green's L-class: Transformation( [ 5, 5, 6, 6, 6, 6 ] )>,
  <Green's L-class: Transformation( [ 5, 5, 6, 6, 6, 7, 7 ] )>,
  <Green's L-class: Transformation( [ 5, 5, 6, 6, 7, 7, 7 ] )>,
  <Green's L-class: Transformation( [ 5, 5, 6, 7, 7, 7, 7 ] )>,
  <Green's L-class: Transformation( [ 5, 6, 6, 6, 6, 6 ] )>,
  <Green's L-class: Transformation( [ 5, 6, 6, 6, 6, 7, 7 ] )>,
   <Green's L-class: Transformation( [ 5, 6, 6, 6, 7, 7, 7 ] )>,
  <Green's L-class: Transformation( [ 5, 6, 6, 7, 7, 7, 7 ] )>,
  <Green's L-class: Transformation( [ 5, 6, 7, 7, 7, 7, 7 ] )>,
  <Green's L-class: Transformation( [ 6, 6, 6, 6, 6, 6 ] )>,
  <Green's L-class: Transformation( [ 6, 6, 6, 6, 6, 7, 7 ] )>,
  <Green's L-class: Transformation( [ 6, 6, 6, 6, 7, 7, 7 ] )>,
  <Green's L-class: Transformation( [ 6, 6, 6, 7, 7, 7, 7 ] )>,
  <Green's L-class: Transformation( [ 6, 6, 7, 7, 7, 7, 7 ] )>,
   <Green's L-class: Transformation( [ 6, 7, 7, 7, 7, 7, 7 ] )>,
  <Green's L-class: Transformation( [ 7, 7, 7, 7, 7, 7, 7 ] )> ]
 generators of S

[ IdentityTransformation, Transformation( [ 2, 2 ] ),
Transformation( [ 2, 3, 3 ] ), Transformation( [ 2, 3, 4, 4 ] ),
  Transformation( [ 2, 3, 4, 5, 5 ] ), Transformation( [ 2, 3, 4, 5, 6, 6 ] ),
  Transformation( [ 2, 3, 4, 5, 6, 7, 7 ] ) ]
gap>

S := Semigroup(Transformation([1, 2, 3, 4, 5, 6, 7]),
> Transformation([2, 2, 3, 4, 5, 6, 7]), Transformation([2, 3, 3, 4, 5, 6, 7]),
> Transformation([2, 3, 4, 4, 5, 6, 7]), Transformation([2, 3, 4, 5, 5, 6, 7]),
> Transformation([2, 3, 4, 5, 6, 6, 7]), Transformation([2, 3, 4, 5, 6, 7, 7]));

S := Semigroup(Transformation([1, 2, 3, 3]), Transformation([1, 2, 3, 2]),
 Transformation([2, 2, 3, 4]),
 Transformation([3, 2, 3, 4]), Transformation([2, 2, 3, 3]),
  Transformation([1, 2, 2, 2]), Transformation([3, 3, 3, 4]),
 Transformation([1, 2, 1, 2]),Transformation([3, 4, 3, 4]),
Transformation([3, 2, 3, 2]), Transformation([4, 3, 3, 4]),
Transformation([1, 2, 2, 1]),Transformation([1, 2, 1, 1]),
Transformation([3, 2, 3, 3]), Transformation([4, 4, 3, 4]),
Transformation([2, 2, 3, 2]));
```