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The Construction and Reconstruction of the Neutrosophic Hom-Groups and Neutrosophic Hom-Subgroups from the Indigenous and Primitive Hom-Group

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Abstract

Hom-groups are the non-associative generalization of a group whose associativity and unitality are twisted by a compatible bijective map. The neutrosophic set is a powerful tool in dealing with incomplete, indeterminate and inconsistent data that exist in the real world. Neutrosophic set is characterized by the truth membership function in the set (T), indeterminacy membership function in the set (I) and falsity membership function in the set (F) where $0 \leq T + I + F \leq 3$. In this work, we have been able to give some introductory entities on the concept of both Hom – groups as well as the neutrosophic Hom – groupsour utmost aim is to construct neutrosophic Hom - group and neutrosophic Hom-subgroupsfrom the already known Hom-groups.

Keywords: Hom-groups, Hom-subgroups, Neutrosophic groups, Neutrosophic subgroups, Neutrosophic hom – groups, Neutrosophic hom – subgroups.
MSC2010: 20A05, 20E34, 20A99.

1 Introduction

In algebra, a common method of investigation is to break up a complex structure into simpler substructures. The hope is that by repeated application of this procedure, one will eventually arrive at structures that are easy to understand. It may then be possible, in some sense, to synthesis these substructures to reconstruct the original one. While it is rare for the procedure just described to be brought to such a perfect state of completion, the analytic synthetic method can yield valuable information and suggest new concepts. Group theory in general, considers how far this can be used. Detailed discussions on general group theories and their various applications involving neutrosophy as well as uncertainty theories could easily be accessible in our references [1–12, 14, 14]. The Hom-Lie algebras including some of its generalizations was believed to have been introduced by Hartwig et.al. This was discovered in one of their seminal papers [13]. In the paper, the deformations of Witt and Virasoro algebras was studied. Meanwhile, Jiang et.al. in [18] studied Hom-Lie algebras and Hom-Lie groups and the discoveries were presented as several results. In [19], Laurent-Gengouxa et.al. introduced the notion of a Hom-group and using their work as the foundation,

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Hassanzadeh [14] took the twisting map to be invertible and then used the invertibility of to study and establish several fundamental properties of the Hom-groups with interesting examples. Specifically, it was established that Lagrange's theorem holds in any finite Hom-group and cosets were used to partition a Hom-group. In [15], Hassanzadeh further studied Hom-groups, their representations and homological algebras. Several interesting results were presented. Based on the work of Hassanzadeh in [14], Liang et.al. in [20] extensively studied Hom-groups and Hom-group actions. In their work, new examples of Hom-groups were provided, further properties of Hom-groups were equally presented. Specifically, the first, second and third isomorphism theorems of Hom-groups were presented. They introduced the concept of Hom-group action and as an application of this concept, they established the first Sylow theorem for Hom-groups. The concept of neutrosophy and neutrosophic set was introduced by Smarandache in [21]. Neutrosophic set is the generalization of fuzzy set introduced by Zadeh in [25] and intuitionistic fuzzy set introduced by Atanassov in [11]. Neutrosophic set is a powerful tool in dealing with incomplete, indeterminate and inconsistent data that exist in the real world. Neutrosophic set is characterized by the truth membership function in the set (T), indeterminacy membership function in the set (I) and falsity membership function in the set (F) where $0 \leq T + I + F \leq 3+$. Vasantha Kandasamy and Smarandache in [24] introduced the concept neutrosophic algebraic structures. In [22], Smarandache introduced and studied (T, I, F)- neutrosophic structures and presented their properties. In [23] Smarandache studied neutrosophic quadruple numbers, refined neutrosophic quadruple numbers, absorbance law, and the multiplication of neutrosophic quadruple numbers. Since the introduction of neutrosophic algebraic structures in [24], many neutrosophic researchers have been working on different neutrosophic algebraic structures and hyper structures such as neutrosophic groups, neutrosophic rings, neutrosophic vector spaces, neutrosophic modules, neutrosophic hypergroups, neutrosophic hyperrings, neutrosophic hypervector spaces, neutrosophic hypermodules, etc. see [1–4, 6–33]. In [30] , some forms of morphism relationship that exists between a neutrosophic Hom-group G(I) and a Hom-group G × G were studied. For more other detailed as well as more comprehensive studies on the entities involving groups subgroups and semi − groups , we implore our eminent readers and researchers to kindler see our references ([34] through [44]).

Definition 1.1. *(see [27]) A Hom-group consists of a set G together with a distinguished member 1 in G, a bijective set map $\alpha : G \to G$, a binary operation $\mu : G \times G \to G$, where these pieces of structure are subject to the following axioms: (1) The product map $\alpha : G \to G$ satisfies the Hom-associativity property $((g), (h, k)) = ((g, h), (k))$.*

Explanations: Since μ is the binary operation of the group, set $\mu =" \bigodot "$ for a general case. We have, the first axiom given by the following: For every $g, h, k, G, \alpha(g) \bigodot (h \bigodot k) = (g \bigodot h) \bigodot \alpha(k)$ We have the following examples.

- Let $\bigodot = +$, the ordinary addition, we have, the first axiom given by: For every $g, h, k, \in G, \alpha(g) + (h + k) = (g + h) + \alpha(k)$.
- Let $\bigodot = x$, the ordinary multiplication, we have, that For every $g, h, k, \in G, \alpha(g)(hk) = (gh)\alpha(k)$.

Here, μ indicates the binary operation under which the group G assumes its closure as a set. Hence, for simplicity sake, the multiplication sign μ may be omitted where necessary.

2. The map α is multiplicative $\alpha(gh) = \alpha(g)\alpha(h)$.
3. The element 1 is called unit and it satisfies the Hom-unitarity conditions $g1 = 1g = \alpha(g)$.
4. For every element $g \in G$, there exists an element $g1 \in G$ such that $gg1 = g1g = 1$.

Without loss of generality, denote the Hom-group by the pair (G, α).

Definition 1.2. *A Hom-group (G, α) is called an idempotent Hom-group if $a^2 = a$.*

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2 Construction of a Neutrosophic Hom–group and Hom–subgroups

Definition 2.1. *Let $H = (H, \mu) = (G, \alpha)$. be any Hom-group and let $\langle H \bigcup I \rangle = \{a+bI \ni a, b \in G\}$. and I is the indeterminate element . Define $NH(G) = (\langle H \bigcup I \rangle, \mu)$. Then, $NH(G)$ is called a neutrosophic Hom–group which is generated by H and I under the binary operation μI is called the neutrosophic element with the property $I^2 = I$. For an integer $n, n + I$, and nI are neutrosophic elements and $0.I = 0.I1$, the inverse of I is not defined and hence does not exist. Any neutrosophic Hom–group $NH(G)$ in which $\forall a, b, \in NH(G), ab = ba$, is said to be commutative.*

For the axioms, we have as follows: Recall that, $g, h, k \in NH(G), g = (g_1 + g_2I), h = (h_1 + h_2I)$, and $k = (k_1 + k_2I)$.

1. The product map. $\alpha : NH(G) \to NH(G)$ satisfies the Hom-associativity property $\mu(\mu(g_1 + g_2I), \mu((h_1 + h_2I), (k_1 + k_2I))) = \mu(\mu((g_1 + g_2I), (h_1 + h_2I)), \alpha(k_1 + k_2I))$.

   $\mu(\alpha(g_1) + \alpha(g_2)I, \mu((h_1 + h_2I), (k_1 + k_2I))) = \mu(\mu((g_1 + g_2I), (h_1 + h_2I)), \alpha(k_1) + \alpha(k_2)I)$.

2. The map α is multiplicative $\alpha((g_1 + g_2I)(h_1 + h_2I)) = \alpha(g_1 + g_2I)\alpha((h_1 + h_2I))$.

   LHS $= \alpha(g_1h_1 + g_1h_2I + g_2h_1I + g_2h_2I)\alpha(a_1 + b1I) = \alpha(a_1) + \alpha(b_1)I, where\alpha(a_1), \alpha(b_1) \in G$.

   RHS $= \alpha(g_1)\alpha(h_1) + \alpha(g_1)\alpha(h_2)I + \alpha(g_2)\alpha(h_1)I + \alpha(g_2)\alpha(h_2)I$.

   $= \alpha(g_1h_1)+\alpha(g_1h_2)I+\alpha(g_2h_1)I+\alpha(g_2h_2)I = \alpha(g_1h_1+g_1h_2I+g_2h_1I+g_2h_2I) \equiv \alpha(a_1+b_1I) = \alpha(a_1) + \alpha(b_1)I$.

   Hence, LHS = RHS.

3. The element 1 is called unit and it satisfies the Hom-unitarity conditions $(g_1 + g_2I)1 = 1(g_1 + g_2I) = \alpha(g_1 + g_2I)$.

4. For every element $(g_1 + g_2I) \in NH(G)$, there exists an element $(g_1 + g_2I)1 \in NH(G)$ such that $(g_1 + g_2I)(g_1 + g_2I)1 = (g_1 + g_2I)1(g_1 + g_2I) = 1$.

   For this to be valid, g_2 must be equal to zero since I^1, the inverse of I is not defined. Hence, this can only work for the neutrosophic Hom–groups $NH(G)$ whose indeterminate is zero. Hence in this case, our operation must therefore exclude multiplication since 1^{-1}, the inverse of 1 does not exist.

Now, given that. $x, y, z \in NH(G) = (\prec H \cup I \succ, ^*)$, where, $x = (a+bI), y = (c+dI)$ and $z = (e+fI)$. Here, the action of α on the element x can be given by. : $\alpha(x)\alpha(a + bI) = \alpha(a) + \alpha(b)I$.

Whence, $\alpha(x^*y) = \alpha([a + bI]^*[c + dI])$ which should be equal to $\alpha([a + bI]^*\alpha([c + dI])$.

Now, LHS is given by $:= \alpha([a+bI]^*[c+dI]) = \alpha(a^*c, [a^*d+b^*c+b^*d]I) = \alpha(a^*c)+\alpha([a^*d+b^*c+b^*d]I). = \alpha(a^*c)+[\alpha(a^*d)+\alpha(b^*c)+\alpha(b^*d)]I = [\alpha(a)^*\alpha(c)+((\alpha(a)^*\alpha(d)+\alpha(b)^*\alpha(c)+\alpha(b)^*\alpha(d))I]$.

And the $RHS = \alpha([a + bI]^*\alpha([c + dI]) = (\alpha(a) + \alpha(b)I)^*(\alpha(c) + \alpha(d)I)$.

$= [\alpha(a)^*\alpha(c) + ((\alpha(a)^*\alpha(d) + \alpha(b)^*\alpha(c) + \alpha(b)^*\alpha(d))I]$. Hence , LHS = RHS.

We now establish the composition of the elements of the neutrosophic Hom – group as follows:

First, we define $x^*y = (a^*c, [a^*d + b^*c + b^*d]I)$. By axiom one, $(\alpha(x), y^*z))$ must be equal to $(x^*y, \alpha(z))$. We have as follows:

$$LHS = (\alpha(x), y^*z)) = (\alpha(a + bI), (c + dI)^*(e + fI))) = (\alpha(a) + \alpha(b)I, (c^*e, [c^*f + d^*e + d^*f]I).$$

$$= [(\alpha(a)+\alpha(b)I]^*[c^*e+(c^*f+d^*e+d^*f)I] = \alpha(a)^*c^*e+[\alpha(b)^*c^*e+\alpha(b)^*c^*f+\alpha(b)^*d^*e+\alpha(b)^*d^*f.$$

.

$$+\alpha(a)^*c^*f + \alpha(a)^*d^*e + \alpha(a)^*d^*f]I. \tag{2.1}$$

And by calculations, the RHS is also given by

$$:= a^*c^*\alpha(e)+[b^*c^*\alpha(e)+b^*c^*\alpha(f)+b^*d^*\alpha(e)+b^*d^*\alpha(f)+a^*c^*\alpha(f)+a^*d^*\alpha(e)+a^*d^*\alpha(f)]I. \tag{2.2}$$

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By using the axiom one, comparing (2.1) and (2.2) componentwise. we observe that LHS = RHS. $(\alpha(x), y^*z))$ must be equal to $= (x^*y, \alpha(z))$.

Just like in the case of the neutrosophic groups, we have the following proposition.

Proposition 2.2. *Suppose that NH(G) is any neutrosophic Hom-group.*

1. *NH(G) in general is not a group.*
2. *NH(G) always contain a group.*

The proof of this proposition is exactly just as we had it for the neutrosophic group. (please see [26] for more details).

Definition 2.3. *Let NH(G) be a neutrosophic Hom-group.*

1. *A proper subset N(A) of NH(G) is said to be a neutrosophic Hom-subgroup of NH(G) if N(A) is a neutrosophic Hom–group. This means that is N(A) contains a proper subset which is a Hom-group.*
2. *N(P) is said to be a pseudo neutrosophic Hom–subgroup of the neutrosophic Hom–group if it does not contain a proper subset which is a Hom-group.*

Example 2.4. *Consider the binary operation given on* $\mathbb{R}$ *by.* $\ni a \boxplus b = a + b + \frac{1}{2}(a + b)$.

This definitely, is a non-associative operation. Hence, $(\mathbb{R}, \boxplus)$ *is not a group. It is very easy to show that* $(\mathbb{R}, \boxplus, \circ, \alpha)$ *is a Hom-group, where* $\alpha : \mathbb{R} \to \mathbb{R}$ *can be defined as* $\alpha(x) = \frac{1}{2}(x)$.

In what follows, we show that $N_H(\mathbb{R}) = (\langle H \cup I \rangle, ^*)$ *is a neutrosophic Hom–group.*

Given that, by the associative axiom, we have that for every $x, y, z \in N_H(\mathbb{R})$, *where for every* $x \in N_H(\mathbb{R}), x = a + bI.$ *, with* $a, b \in \mathbb{R}$ *and* I *is the neutrosophic indeterminate number. We show that :* $\alpha(x) \boxplus (y \boxplus z) = (x \boxplus y \boxplus)\alpha(z)$.

Proof. Let $x = a + Ib, y = c + Id$ and $z = e + If$.

$LHS = \alpha(x) \boxplus (y \boxplus z) = \frac{3}{2}(x) \boxplus (y + z + \frac{1}{2}(y + z)) = \frac{3}{2}(x) + \frac{3}{2}(y) + \frac{3}{2}(z) + \frac{3}{4}(x) + \frac{3}{4}(y) + \frac{3}{4}(z).$
$= \frac{9}{4}(x + y + z) = \frac{9}{4}(a + c + e + [b + d + f]I).$

$RHS = [x \boxplus y] \boxplus \frac{3}{2}(z) = [x + y + \frac{1}{2}(x + y)] \boxplus \frac{3}{2}(z) = \frac{3}{2}(x) + \frac{3}{2}(y) + \frac{3}{2}(z) + \frac{1}{2}(\frac{3}{2}(x) + \frac{3}{2}(y) + \frac{3}{2}(z)).$
$= \frac{9}{4}(x + y + z) = \frac{9}{4}(a + c + e + [b + d + f]I), LHS = RHS.$ □

Example 2.5. *Let the binary operation be given on* $\mathbb{R}$ *by :* $a \oplus b = 1(a + b)$ *where* $k \in \mathbb{C}$. *This is also a non-associative operation. It follows that,* $(\mathbb{R}, \oplus)$ *is not a group. Hence, It can be proved that* $(\mathbb{R}, \oplus, \circ, \alpha)$ *is a Hom-group, where* $\alpha : \mathbb{R} \to \mathbb{R}$ *is defined as* $\alpha(x) = 1(x)$. *We have, by the first axiom for the neutrosophic Hom–group, we are going to have that :* $\alpha(x) \oplus (y \oplus z) = (x \oplus y \oplus)\alpha(z)$.

Proof. Let $x = a + Ib, y = c + Id$ and $z = e + If$ $LHS = \alpha(x) \oplus (y \oplus z) = \frac{1}{k}(x) \oplus (\frac{1}{k}(y + z)) = \frac{1}{k}(\frac{1}{k}(x) + \frac{1}{k}(y + z)) = \frac{1}{k^2}(x + y + z) = \frac{1}{k^2}(a + c + e + [b + d + f]I).$

$RHS = [x \oplus y] \oplus \frac{1}{k}(z) = [\frac{1}{k}(x + y)] \oplus \frac{1}{k}(z) = \frac{1}{k}\frac{(\oplus(x+y)] + \oplus(z))}{k^2} = (\frac{1}{k^2}(x + y + z)) = \frac{1}{k^2}(a + c + e + [b + d + f]I).$

LHS = RHS. □

Example 2.6. *Suppose that* $x^*y = n(\mid x \mid + \mid y \mid), forall x, y, \in \mathbb{R}$, *and any* $n \in \mathbb{N}$ *It can be proved that* $(\mathbb{R}, *, \circ, \alpha)$ *is a Hom-group, where* $\alpha : \mathbb{R} \to \mathbb{R}$ *is defined as* $\alpha(x) = n \mid x \mid$, *and hence, for every* $x, y, z \in N_H(\mathbb{R})$, *where* $x \in N_H(\mathbb{R}) \Longrightarrow x = \{a + bI\}$, *with* $a, b \in \mathbb{R}$ *and* I *is the neutrosophic indeterminate number.*

Example 2.7. *Consider the structure,* $A = (X, \circ)$ *such that :* $x \circ y = \sqrt[n]{(xy)}$ *, for every* $x, y \in X$. *A is a non – associative operation on* $R - 0$. *Hence,* $(R - 0, 1)$ *is not a group. Now, suppose that :* $R - 0 \to R - 0$ *is defined as* $(x) = \sqrt[n]{(x)}$. *it could be shown that* $(R - 0, \circ, 1,)$ *is a Hom–group. First , check that is a bijection on* $R - 0$. *We have as follows:* $(x \circ y) = (\sqrt[n]{xy}) = \sqrt[n]{(\sqrt[n]{(xy)})} = \sqrt[n^2]{xy}$. *Clearly, this shows that is multiplicative.*

However, consider $(x) \circ (y \circ z) = \sqrt[n]{x} \circ \sqrt[n]{yz} = \sqrt[n]{(\sqrt[n]{x})}\, \sqrt[n]{yz} = \sqrt[n]{xyz}.,$

$(x \circ y) \circ (z) = (\sqrt[n]{xy}) \circ \sqrt[n]{z}) = \sqrt[n]{\sqrt[n]{xy}\, \sqrt[n]{z}} = \sqrt[n]{xyz}.,$

and

$(x \circ 1) = \sqrt[n]{x1} = \sqrt[n]{x} = (x) = \sqrt[n]{1x} = (1 \circ x).$

By this , $\circ$ satisfies the Hom–associativity as well as the Hom–identity. Also , $\frac{1}{x}$ is the inverse of every $x \in R - 0$.

Proposition 2.8. *Let $x^*y = \bar{xy}$, the conjugate of the product of $x, y, \forall x, y, \in \mathbb{C}$, the set of complex numbers. Then, $(\mathbb{C},^*, 1, \alpha)$ is a Hom-group, where $\alpha : \mathbb{C} \to \mathbb{C}$ is defined as $\alpha(x) = \bar{x}$ and hence, for every $x, y, z \in NH(\mathbb{C})$, where $x \in NH(\mathbb{C}) \Rightarrow x = a + bI$, with $a, b \in \mathbb{C}$ and I is the neutrosophic indeterminate number, $NH(\mathbb{C})$ is a neutrosophic Hom–group.*

Proof. Observe that $\alpha(x) = \alpha(a+Ib) = a \bar{+} Ib$. Now, since $a, b \in \mathbb{C}$, set $a = a_1+ia_2, and b = b_1+ib_2$.

We have , $\alpha(x) = \alpha(a_1 + ia_2 + I(b_1 + ib_2)) = \alpha((a_1 + Ib_1) + i(a_2 + Ib_2)) = ((a_1 + Ib_1) + i(a_2 + Ib_2)) = ((a_1 + Ib_1) + (a_2 + Ib_2)) = (a_1 + Ib_1) - i(a_2 + Ib_2)$. It is easy to show that : $\alpha(x)(^*y^*z) = (x^*y^*)\alpha(z)$. □

Proposition 2.9. *Given that $\mathbb{Z}_\ltimes = (\mathbb{Z}_\ltimes, 0, +) = (Z, \alpha)$ is a Hom-group and let $\langle \mathbb{Z}_n \bigcup I \rangle = a + bI : a, b \in \mathbb{Z}_n$, (the set of integers modulo n), and I is the indeterminate element. Then, $NH(\mathbb{Z}_\ltimes) = (\mathbb{Z}_\ltimes \bigcup, +)$ is a neutrosophic Hom–group.*

Proof. Let $(a_1 + b_1I), (a_2 + b_2I)$, and $(a_3 + b_3I) \in NH(\mathbb{Z}_\ltimes)$. Then , we show that:

- $(a + bI) + 0 = 0 + (a + bI) = \alpha(a + bI)$, for $a, b \in \mathbb{Z}_n$.
- $\alpha(a_1 + b_1I)^*((a_2 + b_2I)^*(a_3 + b_3I)) = ((a_1 + b_1I)^*(a_2 + b_2I))^*\alpha(a_3 + b_3I).$, for $a_i, b_i \in \mathbb{Z}_\ltimes$ and I , an indeterminate number.

For the first case , we have that : $\alpha(a + bI) = (a + bI) = (a + bI)$.

For the second case, $LHS = \alpha(a_1 + b_1I)^*((a_2 + b_2I)^*(a_3 + b_3I)) = \alpha(a_1 + b_1I)^*((a_2 + a_3) + (b_2 + b_3)I)$.

$= (a_1 + b_1I) + ((a_2 + a_3) + (b_2 + b_3)I) = ((a_1 + a_2 + a_3) + (b_1 + b_2 + b_3)I)$.

$RHS = ((a_1 + b_1I)^*(a_2 + b_2I))^*\alpha(a_3 + b_3I) = ((a_1 + a_2) + (b_1 + b_2)I) + alpha(a_3 + b_3I)$.

$= ((a_1 + a_2 + a_3) + (b_1 + b_2 + b_3)I)$. □

We have the following Cayley Table for the $\mathbb{Z}_2(I)$ and $\mathbb{Z}_3(I)$ Neutrosophic Hom–subgroup.

Table 1: Cayley Table for the $\mathbb{Z}_2^*(1)$

⊕	0	1	1	1 + I
0	0	1	1	1 + I
1	1	0	1 + I	1
1	1	1 +I	0	1
1 + I	1 + I	1	1	0

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Table 2: Cayley Table for the $\mathbb{Z}_2'^{*}(1)$

⊗	0	1	1	1 + I
0	0	0	0	0
1	0	1	1	1 + I
1	0	1	1	0
1 + I	0	1 + I	0	1 + I

Table 3: Cayley Table for the $\mathbb{Z}_3^{*}(1)$

⊕	0	1	2	I	2I	1+ I	2 + I	1 + 2I	2 + 2I
0	0	1	2	I	1+2I	1+I	2+I	1+2I	2+2I
1	1	2	0	1+I	1+2I	2+I	I	2+2I	2I
2	2	0	1	2+I	2+2I	I	1+I	2I	1+2I
I	I	1+I	2+I	2I	0	1+2I	2+2I	1	2
2I	2I	1+2I	2+2I	0	I	I	2	1+I	2+I
1+I	1+I	2+I	I	1+2I	I	2+2I	2I	2	0
2+I	2+I	I	1+I	2+2I	2	2I	1+2I	0	1
1 + 2I	1 + 2I	2+2I	2I	I	1+I	2	0	2+I	I
2+2I	2+2I	2I	1+2I	2	2+I	0	1	1	1+I

Table 4: Cayley Table for the $\mathbb{Z}_3'^{*}(1)$

⊕	0	1	2	I	2I	1+ I	2 + I	1 + 2I	2 + 2I
0	0	0	0	0	0	0	0	0	0
1	0	1	2	1	2I	1+I	2+I	1+2I	2+2I
2	0	2	1	2I	I	2+2I	1+2I	2+I	1+I
I	0	I	2I	I	2I	2I	0	0	1
1+I	0	1+I	2+2I	2I	I	1	2+I	1+2I	1
2+I	0	2+I	1+2I	0	0	2+I	1+2I	2+I	1+2I
1+2I	0	1+2I	2+I	0	0	1+2I	2+1	1+2I	2+I
1 + I	0	1+I	2+2I	2I	I	I	2+I	1+2I	2
2+2I	0	2+2I	1+I	I	2I	2	1+2I	2+I	1

3 Discussion / Explanation

While Tables 1 and 2 are both valid for the Neutrosophic Hom–Group, in Tables 3 and 4, there exist two distinct identities in each case. Hence, we have the following proposition:

Proposition 3.1. *For any $k \in \mathbb{N}$, it is vividly clear that the subgroup, $(\mathbb{Z}_k(I), +, (0, 0))$ is a neutrosophic Hom–Group. For any $k \in \mathbb{N}$, it is vividly clearly that the subgroup clearly, $(\mathbb{Z}_k(I), \otimes, (0, 0))$ would only be a neutrosophic Hom–Group if and only if the identities are not unique.*

Next, we are going to ascertain the validity of the following cayley and the Latin tables for neutrosophic Hom–group in their respective binary operations.

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Table 5: $G(I) = \{1 + I, a + I, b + I, c + I, d + I, e + I\}$

∗	1+I	a+I	b+I	c + I
1+I	1+I	a+I	b+I	c+I
a+I	a+I	1+I	c+I	b + I
b+I	b+I	c+I	1+I	a+I
c + I	c+I	b + I	a+I	1 + I

Table 6: $G(I) = \{1 + I, a + I, b + I, c + I, d + I, e + I\}$

♯	1+I	a+I	b+I	c + I
1	1+I	a+I	b+I	c+I
A	a+I	b+I	c+I	1 + I
B	b+I	c+I	1+I	a+I
C	c+I	1+ I	a+I	b + I

Table 7: $G(I) = \{1 + I, a + I, b + I, c + I, d + I, e + I\}$

∗	1+I	a+I	b+I	c + I	d+I	e+I
1+I	1+I	a+I	b+I	c+I	d+I	e+I
a+I	a+I	b+I	1+I	e+I	c+I	d+I
b+I	b+I	1+I	a+I	d+I	e+I	c+I
c+I	c+I	d+ I	e+I	1+I	a+I	b+I
d+I	d+I	e+ I	c+I	b+I	1+I	a+I
e+I	e+I	c+ I	d+I	a+I	b+I	1+I

Table 8: The Latin square for : $G(I) = \{1 + I, a + I, b + I, c + I, d + I\}$

■	1+I	a+I	b+I	c + I	d+I
1+I	1+I	a+I	b+I	c+I	d+I
a+I	a+I	1+I	d+I	b+I	c+I
b+I	b+I	c+I	1+I	d+I	a+I
c+I	c+I	d+ I	a+I	1+I	b+I
d+I	d+I	b+ I	c+I	a+I	1+I

To prove that the Cayley Tables are valid as neutrosophic Hom–groups, it is necessary to prove that:

1. $\alpha(x + I) = (x + I)(1 + I) = (1 + I)(x + I) = (x + I).$,

   and sufficient to prove that:

2. $\alpha(a + I)[(b + I)(c + I)] = [(a + I)(b + I)]\alpha(c + I)$.

Now, from Tables 3, 4, and 5 , we have the first condition as: $\alpha(x + I) = (x + I)(1 + I) = (1 + I)(x + I) = x + I + xI + I^2 = x(1 + I) + I = x + I$. *(since* $(1 + I)$ *is the identity) for every* $(x + I) \in G(I)$.

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The second case (the sufficient condition) is obviously true. In Table 6, even though the necessary condition is true. i.e. $\alpha(x+I) = (x+I)(1+I) = (1+I)(x+I) = (x+I)$. *For every* $(x+I) \in G(I)$, *it is not true that* $\alpha(a+I)[(b+I)(c+I)] = [(a+I)(b+I)]\alpha(c+I)$. *Hence, the cayley Table 6 does not form a neutrosophic Hom–group.*

Proposition 3.2. *In neutrosophic Hom–Groups, the identities are not unique in general.*

Proof. To prove that the cayley tables are valid as neutrosophic Hom–groups, it is necessary to prove that:

1. $(x+I) = (x+I)(1+I) = (1+I)(x+I) = (x+I)$. And sufficient to prove that:
2. $(a+I)[(b+I)(c+I)] = [(a+I)(b+I)](c+I)$.

Now, from tables 3, 4, and 5 , we have the first condition as:

$\alpha(x+I) = (x+I)(1+I) = (1+I)(x+I) = x+I+xI+I^2 = x(1+I)+I = x+I$. (since $(1+I)$ is the identity) for every $(x+I) \in G(I)$.

The second case (the sufficient condition) is obviously true. In Table 6, even though the necessary condition is true. i.e. $(x+I) = (x+I)(1+I) = (1+I)(x+I) = (x+I)$, for every $(x+I) \in G(I)$. It is not true that;

$(a+I)[(b+I)(c+I)] = [(a+I)(b+I)](c+I)$. Hence, the Cayley Table 6 does not form a neutrosophic Hom–group. □

Definition 3.3. *Let* $M_{m\times n} = (a_{ij})a_{ij} \in K(I)$ *,where* $K(I)$ *is a neutrosophic field. Then* ,$M_{m\times n}$ *is called the neutrosophic matrix of order* $m \times n$.

3.1 The Neutrosophic Hom–Group Representation

Definition 3.4. *Let* $H = A + BI$ *be a neutrosophic square matrix of order n, where P and Q are two n-squares matrices, then H is called an invertible matrix, if and only if there exists an n square matrix* $P = C + DI$, *where* A_1 *and* A_2 *are two n square matrices such that* $HP = PH = T_{n\times n}$, *where* $T_{n\times n}$ *denote the identity matrix of order n.*

Vivid Example: *Clear examples can easily be constructed here, in which case, we set* $HP = (A+BI)(C+DI) = AC + (AD+BC+BD)I$. *Here, such matrices A, B, C, and D exist such that,* $AC = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *and* $(AD+BC+BD) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, *annihilating the entire indeterminate part. All these strictly function o the choices of our matrices entries as required. Similar analogy as well goes for PH*

By the definitions above, there exists neutrosophic matrices representation. Hence, let G(I) in the Cayley Table 5 be represented as:

$$G^{'}(I) = \begin{matrix} 1^{'} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} I,\ a^{'} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} I,\ b^{'} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} I, \\ c^{'} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} I,\ d^{'} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} I,\ e^{'} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} I. \end{matrix}$$

where I is the indeterminacy number, then each of the elements $x \in G^{'}(I)$ *is of the form:* $x = P + QI$ *be a neutrosophic square matrix of order n, where P and Q are two square matrices of order 2. Then,* $G^{'}(I)$ *forms a neutrosophic Hom–group. This is an example of a typical representation for the neutrosophic Hom–Group. In particular, this group representation is otherwise referred to as the faithful representation of the neutrosophic Hom–Group.*

Proposition 3.5. *Suppose that* $N_H(V)$ *is a nonempty proper subset of a neutrosophic Hom-group* $(N_H(G),^*).N_H(V)$ *is a neutrosophic Hom-subgroup of* $N_H(G)$ *if and only if the following conditions hold:*

1. *every* $(a_1, a_2I), (b_1, b_2I), \in N_H(V) \Rightarrow (a_1, a_2I)^*(b_1, b_2I) \in N_H(V)\ \forall\ a, b \in N_H(V)$.
2. *There exists a proper subset S of* $N_H(V)$ *such that* $(S, *)$ *is a neutrosophic Hom - group.*

Definition 3.6. *Let G be any group and x, any element in G. Then, the set $\langle xrangle = \{x^k : k \in \mathbb{Z}\}$ is a subgroup of G. Furthermore, $\langle a \rangle$ is the smallest subgroup of G containing a. By the foregoing definition, if the notation is used, as in the case of the integers under addition, we write $\langle a \rangle = \{na \ni n \in \mathbb{Z}\}$.*

Definition 3.7. *Let $(N_H(G),^*)$ be a neutrosophic Hom- group . For $(a + bI) \in (N_H(G),^*)$, we call $\langle (a + bI) \rangle$ the cyclic neutrosophic Hom - subgroup generated by $(a + bI)$ if $(a + bI)$ is a single element of $(N_H(G),^*)$, and we write $(N_H(G),^*) = \langle (a + bI) \rangle$, then $(N_H(G),^*)$ is a neutrosophic Hom cyclic group.*

Example 3.8. *Consider a neutrosophic Hom - subgroup : $A = \{0, I\} \subset \{0, 1, I, 1+I\} = (_2(I), +, (0, 0)) = (N_H(G),^*)$. Clearly, $\{I\}$ generates $(NH(G),^*)$. Hence, we can write $(NH(G),^*) = \{\langle I \rangle\}$. This means that $\langle I \rangle$ generates $(N_H(G),^*)$.*

Proposition 3.9. *Every neutrosophic Hom - cyclic group is abelian.*

Proposition 3.10. *Every neutrosophic Hom - subgroup of a neutrosophic Hom - cyclic group is cyclic.*

Proposition 3.11. *Let $N(H)$ be a nonempty proper subset of a neutrosophic Hom-group $((N_H(G),^*),^*)$. $N_H(V)$ is a pseudo neutrosophic Hom - subgroup of $(N_H(G),^*)$ if and only if the following conditions hold:*

1. *$a, b \in N_H(V)$ implies that $a^*b \in N_H(V) \forall a, b \in N_H(V)$.*
2. *$N_H(V)$ does not contain a proper subset A such that $(A,^*)$ is a group.*

Definition 3.12. *Let $N_H(V)$ and $N_H(U)$ be any two neutrosophic Hom - subgroups of a neutrosophic Hom - group $N_H(G)$. The product of $N_H(V)$ and $N_H(U)$ denoted by $N_H(V).N_H(U)$ is the set $N_H(V).N_H(U) = \{vu : v \in N_H(V), u \in N_H(U)\}$.*

Definition 3.13. *Let $N_H(V)$ and $N_H(U)$ be any two pseudo neutrosophic Hom - subgroups of a neutrosophic group $N_H(G)$. The product of $N_H(V)$ and $N_H(U)$ denoted by $N_H(V).N_H(U)$ is the set $N_H(V).N_H(U) = \{vu : v \in N_H(V), u \in N_H(U)\}$.*

Definition 3.14. *Let $N_H(G)$ be a neutrosophic Hom-group. The order of $(N_H(G),^*)$ denoted by $\circ((N_H(G),^*)$ or $|(N_H(G),^*)|$ is the number of distinct elements in $(N_H(G),^*)$. Suppose that the order $\circ((N_H(G),^*))$ of the neutrosophic Hom - group $(NH(G),^*)$ is finite. Then , $(N_H(G),^*)$ is known as a finite neutrosophic Hom-group and infinite neutrosophic Hom-group if $\circ((N_H(G),^*)) \nless \infty$.*

4 Subgroups of Neutrosophic Hom-group

Consider the $(\mathbb{Z}_2(I), +, (0, 0)), (\mathbb{Z}_3(I), +, (0, 0)) \in (\mathbb{Z}_n(I), +, (0, 0)) \subset (N_H(G),^*)$. There exist the following chains of subgroups: $\{0\} = \langle 0 \rangle \subset (\mathbb{Z}_2(I), +, (0, 0))$.

$\{0\} \subset \{0, 1\} = \langle 1 \rangle \subset \{0, 1, I, 1 + I\} = \langle 1, I \rangle \subset (\mathbb{Z}_2(I), +, (0, 0))$.

$\{0\} \subset \{0, I\} = \langle I \rangle \subset \{0, 1, I, 1 + I\} = \langle 1, I \rangle \subset (\mathbb{Z}_2(I), +, (0, 0))$.

$\{0\} \subset \{0, 1 + I\} = \langle 1 + I \rangle \subset \{0, 1, I, 1 + I\} = \langle 1, I \rangle \subset (\mathbb{Z}_2(I), +, (0, 0))$.

$\{0\} = \langle 0 \rangle \subset (\mathbb{Z}_3(I), +, (0, 0))$.

$\{0\} \subset \{0, 1\} = \langle 1 \rangle \subset \{0, 1, 2, I, 2I, 1 + I, 1 + 2I, 2 + I, 2 + 2I\} = \langle 1, I \rangle \subset (\mathbb{Z}_3(I), +, (0, 0))$.

$\{0\} \subset \{0, 1, 2\} = \langle 1 \rangle \subset \{0, 1, 2, I, 2I, 1 + I, 1 + 2I, 2 + I, 2 + 2I\} = \langle 1, I \rangle \subset (\mathbb{Z}_3(I), +, (0, 0))$.

$\{0\} \subset \{0, I, 2I\} = \langle 1 \rangle \subset \{0, 1, 2, I, 2I, 1 + I, 1 + 2I, 2 + I, 2 + 2I\} = \langle 1, I \rangle \subset (\mathbb{Z}_3(I), +, (0, 0))$.

$\{0\} \subset \{0, 1 + I, 2 + 2I\} = \langle 1 + I \rangle \subset \{0, 1, 2, I, 2I, 1 + I, 1 + 2I, 2 + I, 2 + 2I\} = \langle 1, I \rangle \subset (\mathbb{Z}_3(I), +, (0, 0))$.

$\{0\} \subset \{0, 2+I, 2+2I\} = \langle 1+2I \rangle = \langle 2+I \rangle \subset \{0, 1, 2, I, 2I, 1+I, 1+2I, 2+I, 2+2I\} = \langle 1, I \rangle \subset (\mathbb{Z}_3(I), +, (0, 0))$.

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Proposition 4.1. *Let $M(I)$ be the maximal subgroup for $(\mathbb{Z}_n(I), +, (0,0)) \subset (NH(G), ^*)$. Then , there exist central normal series for the neutrosophic Hom - Group series ending in $M(I)$ series ending in $(\mathbb{Z}_n(I), +, (0,0)) \subset (NH(G), ^*)$.*

Theorem 4.2. *Let $(\mathbb{Z}_n(I), +, (0,0))$ be a neutrosophic Hom – group then:*

1. *The order of $(\mathbb{Z}_k(I), +, (0,0))$ denoted by: $|\mathbb{Z}_k(I)| = k^2$ for $0 \leq k \leq n \in N$.*
2. *The number of the distinct neutrosophic Hom – subgroup is a function of $(2k+1)$.*
3. *The number of maximal subgroups of $(\mathbb{Z}_k(I), +, (0,0)$ is given by $(k-1)$.*

4.1 The Subgroup Lattices of the Neutrosophic Hom–group

Theorem 4.3. *Let $NH(G)$ be a neutrosophic Hom - group and F the family of all subgroups of $NH(G)$. Define a relation $\mathbb{R}$ on F by $NH(U)\mathbb{R}NH(V)$ if $NH(U) \leq NH(V)$. Then, F form a neutrosophic Hom - subgroup lattice for the group $NH(G)$.*

Lemma 4.4. *Let $NH(G)$ be a neutrosophic Hom-group and $NH(V)$, a neutrosophic Hom-subgroup of $NH(G)$. Define the relation $\mathbb{R}$ on $NH(G)$ by $a\mathbb{R}b$ if $ab^{-1} \in NH(V)$. Then $\mathbb{R}$ is an equivalence of relation on $NH(G)$.*

Proof.

1. For $a \in NH(G), aa^{-1} = e \in NH(G)$, since $NH(V)$ is a neutrosophic Hom - subgroup. So, $a\mathbb{R}a$. i.e $\mathbb{R}$ is reflexive.
2. Also, suppose $a\mathbb{R}b$ i.e $ab^{-1} \in NH(G)$, then $(ab^{-1})^{-1} = ba^{-1} \in NH(V)$. So, $b\mathbb{R}a$. This is symmetric property.
3. Suppose $a\mathbb{R}b$ and $b\mathbb{R}c$, then $ab^{-1} \in NH(V), bc^{-1} \in NH(V), ac^{-1} = ab^{-1}bc^{-1} \in NH(V)$ since closure property holds in $NH(V)$. So $a\mathbb{R}c$ i.e $\mathbb{R}$ is transitive.

Hence $\mathbb{R}$ is an equivalence relation. □

Proposition 4.5. *Let $G(I)$ be a finite abelian neutrosophic Hom - group and $|G(I)| = n = P_1^{r_1}P_2^{r_2}...P_k^{r_k}$, the decomposition of its order into prime power factors. If $G(I) = G(I)p_1 \oplus G(I)p_2 \oplus ... \oplus G(I)p_k$ is the corresponding primary decomposition, then, denoting by $L(G(I))$ the subgroup lattice of $G(I), L(G(I)) \simeq L(G(I)p_1) \times L(G(I)p_2) \times \cdots \times L(G(I)p_k)$, the direct product of the corresponding subgroup lattices (see [28, 29]). We denote by $N(G(I))$ the number of subgroups of the group G. Hence $N(G(I)) = \prod_{i=1}^{k} N(G(I)_{p_i})$ and our counting problem is reduced to p-groups.*

Definition 4.6. *[7] Let F be a field and let A be an F - vector space which is also a ring with identity . suppose that for all cF and $x, yA, (cx)y = c(xy) = x(cy)$. Then, A is an F – algebra.*

Definition 4.7. *: Suppose that G(I) is a finite neutrosophic Hom – group. Set $F[G(I)]$ is the set of "formal" sums $_(gG(I))a_g g|a_g F$. The structure of an F- vector space is given to $F[G(I)]$ and the elements of $F[G(I)]$ for which $ag = 1 + 0I = 1$ and $ah = 0 + 0I = 0 if h = h_1 + h_2 I g = g1 + g2I$ is identified with $g = g_1 + g_2 I G(I)$. This identification embeds G(I) into $F[G(I)]$. This makes the neutrosophic Hom group $G(I)$ a base for $F[G(I)]$.*

Definition 4.8. *[21] Let A be an F- algebra. A representation of A is an algebra homomorphism $: A \to M_n(F)$. The integer n is the degree of.*

Example 4.9. *Let $A = 1, -1, i, -i$. Then for the quaternion neutrisophic Hom – group we have the following matrix representations:*

$1 \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 1 \to \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} - i \to \begin{pmatrix} (1+aI) & 0 \\ 0 & -(1+aI) \end{pmatrix} - i \to$
$\begin{pmatrix} -(1+aI) & 0 \\ 0 & (1+aI) \end{pmatrix}$ *where $a \in \mathbb{C}$.*

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Two representations X, η of degree n are said to be similar if there exists P , a non singular matrix of order n such that $X(a) = P - 1\eta(a)P$ for all a in A. The similarity property is an equivalent relation on representation. Also, if h is a representation of degree n and P is any non-singular matrix of order nxn.Then, the formular $X(a) = P - 1\eta(a)P$ defines a new representation X is said to be in reduced form and the one similar to X is reducible.

Example 4.10. 1. *Let $G(I) \subset SL(n, \mathbb{R})(I)$, the special linear neutrosophic matrix of order n. Then, $G(I)$ is a neutrosophic Hom–group. The center of $G(I)$ given by : $C_{G(I)} = A \in G(I) : AX = XA \; \forall XA \in G(I)\}$ is not trivial. i.e. there exists, not only the identity , which commutes with every other elements in $G(I)$. In particular, let $A \in G(I)$ be a neutrosophic Hom matrix such that: $A = \begin{pmatrix} a+bI & c+dI \\ e+fI & g+hI \end{pmatrix}$. Now, since A is a special linear neutrosophic matrix, $(a+bI)(g+hI) = (c+dI)(e+fI)+1$. implies $ag - ce = 1$, and $[ah+bg+bh]I = [cf+de+df]I$. implies $ah+bg+bh-cf-de-df = 0 \; \forall \; a,b,c,d,e,f,g,h \in \mathbb{R}$ or $\mathbb{C}$. (The possibilities here, also depend on our choices of the real or the complex numbers).*

2. *Let $G(I) = \left\{ g \in G(I) \mid g = \begin{pmatrix} 1 & (a+bI) & (c+dI) \\ 0 & 1 & (e+fI) \\ 0 & 0 & 1 \end{pmatrix} \right.$ $\forall \; a,b,c,d,e,f \in \mathbb{R} \;\; \mathbb{C}$, the Heisenberg neutrosophic group. Then, $G(I)$ is neutrosophic Hom-group.*

   Here, for the bijective map, we have that: for $g \in G(I)(g) = 1g = g1$ as required, and in this case, equal to g.

3. *Let $G(I) = \left\{ g \in G(I) | g = \begin{pmatrix} (a+bI) & 0 \\ 0 & (c+dI) \end{pmatrix} \right.$ $\forall \; a,b,c,d \in \mathbb{R}$ or $\mathbb{C}$. Then , if $(a+bI)^3 = (c+dI)^3 = 1$, then for some $a,b,c,d \in \mathbb{R}$ or $\mathbb{C}$, then, $G(I)$ form a neutrosophic Hom–Group.*

Example 4.11. *if $a = c = 1, b = \left(\frac{-3+i\sqrt{2}}{2}\right)$, and $d = \left(\frac{-3-i\sqrt{3}}{2}\right)$, then $G(1) = \langle A, B \rangle = \{1, A.B \mid A^2 = B^2 = AB = BA = 1, B^2 = A = B^{-1}, A^2 = B = A^{-1}\}$.*

Let $G(I) = \left\{ g \in G(I) | g = \begin{pmatrix} (a+bI) & 0 \\ 0 & (c+dI) \end{pmatrix} \right.$ $\forall \; a,b,c,d \in \mathbb{R}$ or $\mathbb{C}$.

Then if $(a+bI)^2 = (c+dI)^2 = 1$, then for some $a,b,c,d \in \mathbb{R}$ or $\mathbb{C}$, then, $G(I)$ form a neutrosophic Hom–Group.

Example 4.12. *For some $a,b,c,d \in \mathbb{R}$ or $\mathbb{C}, G(I) = \langle A, B \rangle = \{1, A, B, AB\}, 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} A = \begin{pmatrix} (1-2I) & 0 \\ 0 & (1-2I) \end{pmatrix} B = \begin{pmatrix} (1-2I) & 0 \\ 0 & (-1+2I) \end{pmatrix}$ and $AB = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.*

For the example cases 3 and 4 above, the bijective maps can as well be derived analogously.

Theorem : Let $G(I)$ be a group of invertible neutrosophic matrices of order n. Then , since the associative property is satisfied in the neutrosophic matrices, the following statements are true: $(g) = g.1 = 1.g = g$, where 1 is the identity matrix of order n and $g \in G(I)$. Let $x, y, z \in G(I)$. Then , $\alpha(x).[y.z] = [x.y].\alpha(z)$. Hence , $G(I)$ is a neutrosophic Hom – group . The center of $G(I)$ is the set of all the matrices pairs $A, B \in G(I)$ such that $AB = BA$ that is those neutrosophic matrices that commute.

4.2 The Construction of the Quaternion Neutrosophic Hom–group

Recall that the quaternion group of order eight is a hypercomplex number given by: $a_o + a_1 i + a_2 j + a_3 k$, where the coefficients $a_o, a_1 i + a_2, a_3$ are real numbers and the symbols i, j, k, satisfy the relations $i^2 = j^2 = -1, ij = -ji = k$. This also could equivalently be expressed as : $i^4 = 1, i^2 = j^2, ji = i^3 j$. The existence of this particular group is actually confirmed by its faithful matrix representation .

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Now suppose that $x = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *, it is very easy to verify that these matrices satisfy the relation given by :* $x^4 = 1, x^2 = y^2, yx = x^3y$.

Hence, the quaternion group of order eight has an appropriate matrix representations which can be expressed as:

$Q_8 = x, y. \mid x^4 = 1, x^2 = y^2, yx = x^3y\} = 1, x, x^2, x^3, y, xy, x^2y, x^3y\}$. *This group is non – abelian , since there exist* A, B*, such that* $ABBA$ *for some* $A, B \in Q_8$.

4.3 The Construction of the Matix Representation of the Quaternion Neutrosophic Hom–group

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In order to construct a quaternion neutrosophic group, we seek two neutrosophic numbers $(a + bI), (c + dI)$ *from a neutrosophic group* $G(I)$ *, where* I *is the indeterminacy number and* $a, b, \in \mathbb{R}$ *or* $\mathbb{C}$ *. So , let the equivalence of* $\sqrt{-1}$ *and* 1 *be* $(a + bI)$ *and* $(c + dI)$ *respectfully . By the defining structure of quaternion group , we are going to have that* $x = \begin{pmatrix} -(a+bI) & 0 \\ 0 & -(a+bI) \end{pmatrix}, y = \begin{pmatrix} 0 & (c+dI) \\ (c+dI) & 0 \end{pmatrix}$.

Now , because $x^2 = y^2, x^2 = \begin{pmatrix} (a+bI)^2 & 0 \\ 0 & (a+bI)^2 \end{pmatrix}, y^2 = \begin{pmatrix} (c+dI)^2 & 0 \\ 0 & (c+dI)^2 \end{pmatrix}$,

$\Rightarrow x^2 = y^2 = \begin{pmatrix} (a+bI)^2 & 0 \\ 0 & (a+bI)^2 \end{pmatrix}, = \begin{pmatrix} (c+dI)^2 & 0 \\ 0 & (c+dI)^2 \end{pmatrix}, = (a + bI)^2 = (c + dI)^2$

$\Rightarrow (a + bI)^2 = (c + dI)^2, i.e. a = c, b = d$.

Hence let $x = \begin{pmatrix} (a+bI) & 0 \\ 0 & (a+bI) \end{pmatrix}$ *, Then* $\begin{pmatrix} y = 0 & (c+dI)^2 \\ (c+dI)^2 & 0 \end{pmatrix}$*,. Now since* $x^4 = 1$ *we must have that* $x = \begin{pmatrix} (a+bI)^4 & 0 \\ 0 & (a+bI)^4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (a + bI)^4 = 1$.

By using the binomial expansion method , we have : $a^4 + [4a^3b + 6a^2b^2 + 4ab^3 + b^4]I$*. Equating the respective components , we have :* $a^4 = 1$ *, and* $4a^3b + 6a^2b^2 + 4ab^3 + b^4 = 0$.

$a^4 = 1 \Rightarrow a^4 - 1 = 0 \Rightarrow (a^2 - 1)(a^2 + 1) = 0$ *or* $(a - 1)(a + 1)(a - i)(a + i) = 0$ *and* $4a^3b+6a^2b^2+4ab^3+b^4 = 0$ *We have that:* $a = \{1, -1, i, -i\}$*. Now, to solve* $4a^3b+6a2b2+4ab3+b4 = 0$ *, we are going to have four distinct cases of possibilities.*

CASE 1: Set $a = 1$*, we have that* $4b + 6b^2 + 4b^3 + b^4 = 0$*. or* $b^4 + 4b^3 + 6b^2 + 4b = 0 \Rightarrow b(b^3 + 4b^2 + 6b + 4) = 0$ *. But* $b \neq 0$*. Hence, we must have :* $b^3 + 4b^2 + 6b + 4 = 0$*. Recall from [6] that if* $x^3+Ax^2+Bx+C = 0$ *is a cubic monic polynomial the coefficients.* $A, B,$ *and* C *belong to the field* K *of characteristic zero . Now , if* $f(x) = x^3 + Ax^2 + Bx + C$*, then ,* $f(x - \frac{1}{3}A) = x^3 - px - q$*, where* $p = \frac{1}{3}A^2 - B$ *, and* $q = \frac{1}{3}BA - \frac{2}{27}A^3 - C$*. We now attempt to solve the cubic polynomial* $x^3 - px - q$*, where* p *and* q *belong to* K*. So , let* $f(x) = x^3 - px - q$ *and* u *and* v *be elements of some splitting field for* f *over* K *. Then ,* $f(u + v) = u^3 + v3 + (3uv - p)(u + v) - q$ *. Suppose that* $3uv = p$ *. Then ,* $f(u+v) = u^3 + \frac{p^3}{(27u^3)} - q$ *. Thus ,* $f(u+\frac{p}{(3u)}) = 0$ *iff* u^3 *is a root of the quadratic polynomial* $x^2 - xq + \frac{p^3}{27}$*. Now , the roots of this quadratic polynomial are :* $\frac{q}{2} \pm \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}$*, and the product of these roots is* $\frac{p^3}{27}$*. Thus, if one of these roots is equal to* u^3*, then the other is equal to* v^3*, where* $v = \frac{p}{3u}$*. It therefore implies that the roots of the cubic polynomial* f *are:*

$\sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}} + \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}}$*, where, the two cube roots must be chosen so as to ensure that their product is equal to* $\frac{1}{3}p$*. This implies that the cubic polynomial* $x^3 - px - q$ *spits over the field* $K(\mu, \tau, \phi)$ *and* $\mu^2 = \frac{1}{4}q^2 - \frac{1}{27}p^3$ *and* $\tau^3 = \frac{1}{2}q + \mu$*. Also ,* ϕ *must satisfy* $\phi^3 = 1$ *but* $\phi \neq 1$ *. Thus the roots of the polynomial in this field of extension are given by:* $\alpha, \beta,$ *and* γ *and that* $\alpha = \tau + \frac{p}{3\tau}, \beta = \phi\tau + \phi^2\frac{p}{3\tau}$ *, and* $\gamma\phi^2\tau + \phi^3\frac{p}{3\tau}$.

Now, our polynomial is : $b^3+4b^2+6b+4=0$. *if this is compared with* $x^3+Ax^2+Bx+C=0$ *is a cubic monic polynomial the coefficients.* $A, B,$ *and* C *belong to the field* K *of characteristic zero . Now, if* $f(x)=x^3+Ax^2+Bx+C$, *then,* $f(b-\frac{1}{3}.4)=b^3-pb-q$, *where* $p=\frac{1}{3}.4^2-6$ *, and* $q=\frac{1}{3}.6.4-\frac{2}{27}4^3-4.\Rightarrow p=\frac{16}{3}-6=\frac{-2}{3}$ *and* $q=\frac{1}{3}6.4-\frac{2}{27}4^3-4=\frac{24}{3}-\frac{128}{27}-4=\frac{-20}{27}$.

The roots of this quadratic polynomial are now : $\frac{-10}{27}\pm\sqrt{\frac{100}{27x27}+\frac{8}{27x27}}=\frac{-10}{27}\pm\sqrt{\frac{100}{729}+\frac{8}{729}}=\frac{-10}{27}\pm\sqrt{\frac{108}{729}}=\frac{-10}{27}\pm 6\sqrt{3}27$ *, and the product of these roots is* $\frac{-8}{27x27}=\frac{-8}{729}$. *Thus, if one of these roots is equal to* u^3, *then the other is equal to* v^3, *where* $v=\frac{p}{3u}$.

It therefore implies that the roots of the cubic polynomial f are: $\sqrt[3]{\frac{q}{2}+\sqrt{\frac{q^2}{4}-\frac{p^3}{27}}}+\sqrt[3]{\frac{q}{2}-\sqrt{\frac{q^2}{4}-\frac{p^3}{27}}}$ *, where , the two cube roots must be chosen so as to ensure that their product is equal to* $\frac{1}{3}p=\frac{1}{3}(\frac{-2}{3})=\frac{-2}{9}$.

So , $\mu^2=\frac{1}{4}q^2-\frac{1}{27}p^3\mu=\pm\frac{2\sqrt{3}}{9}$ *and* $\tau^3=\frac{1}{2}q+\mu=\frac{-10}{27}\pm\frac{2\sqrt{3}}{9}\tau=\sqrt[3]{\frac{-10}{27}}\pm\frac{2\sqrt{3}}{9}$ *Also ,* ϕ *must satisfy* $\phi^3=1$ *but* $\phi\neq 1\Rightarrow\phi^2+\phi+1=0$, *i.e* $\phi=(-1\pm\frac{i\sqrt{3}}{2})$. *Thus the roots of the polynomial in this field of extension are given by:* α,β *and* γ *and that* $\alpha=\tau+\frac{p}{3\tau}=\sqrt[3]{\frac{-10}{27}\pm\frac{2\sqrt{3}}{9}}+\frac{\frac{-2}{3}}{3\sqrt[3]{\frac{-10}{27}\pm\frac{2\sqrt{3}}{9}}}$,

$\beta=\left(\frac{-1\pm i\sqrt{3}}{2}\right)\left(\sqrt[3]{\frac{-10}{27}\pm\frac{2\sqrt{3}}{9}}+\left(\frac{-1\pm i\sqrt{3}}{2}\right)^2\left(\frac{\frac{-2}{3}}{3\tau}\right)\right.$, *and*

$\gamma=\left(\frac{-1\pm i\sqrt{3}}{2}\right)^2\sqrt[3]{\frac{-10}{27}\pm\frac{2\sqrt{3}}{9}}+\left(\frac{-1\pm i\sqrt{3}}{2}\right)^3\frac{\frac{-2}{3}}{3\sqrt[3]{\frac{-10}{27}\pm\frac{2\sqrt{3}}{9}}}$.

Hence, $b=\{\alpha,\beta,\gamma\}$ *as defined in each respective cases. We have that for the CASE 1,* $a=1\Rightarrow$ *that the quaternion neutrosophic group is given by:*

$$G(I)=\begin{cases} x=\begin{pmatrix}(1+\alpha I) & 0\\ 0 & -(1+\alpha I)\end{pmatrix} \text{ and } y=\begin{pmatrix}0 & (1+\alpha I)\\ (1+\alpha I) & 0\end{pmatrix}\\ x=\begin{pmatrix}(1+) & 0\\ 0 & -(1+\beta I)\end{pmatrix} \text{ and } y=\begin{pmatrix}0 & (1+\beta I)\\ (1+\beta I) & 0\end{pmatrix}\\ x=\begin{pmatrix}(1+\gamma I) & 0\\ 0 & -(1+\gamma I)\end{pmatrix} \text{ and } y=\begin{pmatrix}0 & (1+\gamma I)\\ (1+\gamma I) & 0\end{pmatrix}\end{cases}$$

Theorem 4.13. *$G(I)$ as define above is a quaternion neutrosophic Hom – group*

Proof. The proof follows from the fact that the two axiom namely : The alpha axiom as well as the second The twisted associative axiom are both satisfied. □

Remark 4.14. *Other three cases i.e.* $a=\{-1,i,-i\ni i=\sqrt{-1}$ *can as well be used to find the respective corresponding values of b so as to construct three other distinct cases of the existence of the quaternion neutrosophic Hom – groups.*

By this analysis , the total number of the quaternion neutrosophic Hom – group TYPES that can be formed using this process and method is 12.

$$x=\begin{pmatrix}(1+\alpha I) & 0\\ 0 & -(1+\alpha I)\end{pmatrix} \text{ and } y=\begin{pmatrix}0 & (1+\alpha I)\\ (1+\alpha I) & 0\end{pmatrix}$$
$$x=\begin{pmatrix}(1+\alpha I) & 0\\ 0 & -(1+\alpha I)\end{pmatrix} \text{ and } y=\begin{pmatrix}0 & (1+\alpha I)\\ (1+\alpha I) & 0\end{pmatrix}$$
$$x=\begin{pmatrix}(1+\alpha I) & 0\\ 0 & -(1+\alpha I)\end{pmatrix} \text{ and } y=\begin{pmatrix}0 & (1+\alpha I)\\ (1+\alpha I) & 0\end{pmatrix}.$$

In the same vein the defining relations can be verified that,

$G(I)=Q_8(I)=\{x,y.\ni x^4=1,x^2=y^2,yx=x^3y=1,x,x^2,x^3,y,xy,x^2y,x^3y\}$.

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5 The Construction of the Dihedral Neutrosophic Hom–Group

. Recall that the classical dihedral group of order eight has the structure given by:

$D_8 = \langle a, b \ni a^4 = 1, b^2 = 1, bab^{-1} = a^{-1} \rangle$ *. So, let* $a + bI \in G(I)$ *such that* $(a + bI)4 = 1$*. Write* $(a + bI)^4 = (a + bI)^2(a + bI)^2 = (a^2 + [2ab + b^2]1)^2 = a^4 + [2a^2(2ab + b^2) + (2ab + b^2)^2]1$

$= a^4 + [(2ab + b^2)(2a^2 + 2ab + b^2)]1 = 1, a^4 = 1a = 1, -1, i, -i$ *and* $b = -2ab = \{2, -2, 2i, -2i\}$*. We have that* $G(I) = \langle i, -1 - 2I \ni i^4 = 1, (-1 - 2I)^2 = 1, (-1 - 2I)i(-1 - 2I)^(- 1) = i^(- 1) \rangle$*.* $= \{1, -1, i, -i, -1+2I, -1-2I, i+2i, -i-2iI\}$*. Remarks: Observe that the* $G(I) = D_8$ *constructed is abelian. This actually follows from the fact that the group generators* $a, b \in \mathbb{R} or \mathbb{C}$ *The construction of the quaternion neutrosophic Hom–group.*

Proposition 5.1. *The following are equivalent :* a, b *generate* $G(I) = Q_8$ *such that* $a, b \in \mathbb{R}$ *or* $\mathbb{C}$ $G(I) = Q_8$ *is abelian* $G(I) = Q_8 = D_8$*.*

Conclusion *We have been able to successfully construct hom – groups, as well as their counterpart hom – subgroups. Hence, this has paved ways for further reconstructions of other non – abelian hom – groups such as the dihedral as well as the quaternion hom – groups.*

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