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Approximate Analytical Solution of a Class of Singular Differential Equations with Dirichlet Boundary Conditions by the Modified Adomian Decomposition Method

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Abstract

In this paper, the difficulty associated with the numerical solution of a class of singular differential equation with dirichlet-boundary conditions is considered and solved by the Modified Adomian Decomposition Method (MADM), based on a new operator propose to remove its singularity . The new scheme is tested for some examples and the obtained results present. These results reveal the suitability and efficiency of the propose method for this class of problem especially when comparisons is made with the exact solution and other techniques in the literature.

Keywords: Modified Adomian Decomposition Method, Lane-Emden Type Equations, Singularity, Dirichlet Boundary Conditions.
MSC2010: 65L10

1 Introduction

In recent years, the analytic and numerical treatment of singular differential equations with Dirichlet-boundary conditions has always been a difficult and challenging task due to the singularity behaviour that occurs at a point. This paper is concern with a class of singular differential equations with Dirichlet-boundary conditions of the form

$$y^{''} + \frac{2}{x}y^{'} + ky = g(x) + N(y), \quad 0 \leq x < 1 \tag{1.1}$$

subject to the boundary conditions

$$y(0) = \alpha_1, \quad y(b) = \alpha_2, b \neq 0$$

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where $g(x)$ is continuous function on (0, 1], N is a nonlinear operator and k, α_1, α_2 and b are real constants. Problems of the form (1.1) have attracted the attentions of many mathematicians and physicists. These problems are frequently in the field of science and engineering, especially in fluid and quantum mechanics, optimal control problems, chemical reactor theory, aerodynamics, reaction-diffusion process, geophysics just to mention a few. Various methods have been proposed by many researchers for the numerical singular differential equation of the form (1.1). Abu-Zaid and El-Gebeily [1] had earlier solved singular two point boundary value problem using finite difference approximation, variational iteration method was used by Junfeng [2] to investigate the solution of equation (1.1). Moreover, other methods such as cubic splines by Ravi Kanth and Reddy [3], Sine-Galerkin method and Homotopy perturbation method by Al-Khaled [4] has given a general study to construct the exact and series solution of singular two point boundary value problems. Further work on numerical solution of equation (1.1) by differential methods has been carried out in several papers and monographs. (see [5] - [18]). The existence and uniqueness of solution of equation (1.1) is discussed in ([1], [7]) by Abu-Zaid and El-Grebeily, Inc, Ergut and Cherrault respectively.

The decomposition method introduced by George Adomian at the beginning of 1980s, has received immense attention in the past two decades. Adomian ([19], [20]), asserts that the decomposition method provides an efficient and convenient method for generating approximate series solution to a wide class of differential equations which converges. Ever since the advent of ADM, various modifications, different ways of obtaining the Adomian polynomials and its applications in a large variety of mathematical and physical problems involving (ordinary or partial) differential, integral, integro-differential, fractional differential, algebraic and system of such equations has been investigated by Adomian and many other authors. see ([19] - [27]). The convergence of ADM has also been extensively discussed by various authors in their papers, Charrault ([28] - [35]) and his co-workers (specially Abbami) first obtained the convergence of the techniques, Hossein and Nasah Zadeh [36], Adomian and Rach [37] introduced the phenomenum of noise terms which accelerates the convergence of the series solution by Adomian decomposition method. This was further investigated and Wazwaz [?] proposed the concept of effective noise term which further strengthen the findings of Adomian and Rach [37].

In this paper, a new and reliable modification of Adomian decomposition based on a new differential operator is proposed which can be used for linear and nonlinear ordinary differential equations. The main idea of the method is to create a canonical form containing all boundary conditions so that the zeroth component is explicitly determined without additional calculations and all other components of the series are easily determined.

2 Analysis of the Modified method

In an operator form, Equation (1.1) can be written as

$$Ly = g(x) + N(y) \tag{2.1}$$

where the differential operator L is given by

$$L(.) = \frac{1}{x\cos\sqrt{k}x}\frac{d}{dx}\cos^2\sqrt{k}x\frac{d}{dx}\frac{x}{\cos\sqrt{k}x}(. \tag{2.2}$$

The inverse operator L^{-1} is therefore considered as a two-fold integral operator, becomes:

$$L^{-1}(.) = \frac{\cos\sqrt{k}x}{x}\int_a^x \cos^{-2}\sqrt{k}x\int_0^x x\cos\sqrt{k}x(.). \tag{2.3}$$

By operating L^{-1} on (2), we have

$$y(x) = \phi(x) + L^{-1}(g(x)) + L^{-1}(N(y)), \tag{2.4}$$

such that

$$L(\phi(x)) = 0.$$

The Adomian decomposition method introduce the solution $y(x)$ and the nonlinear function $N(y)$ by infinite series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{2.5}$$

and

$$N(y) = \sum_{n=0}^{\infty} A_n, \tag{2.6}$$

where the components $y_n(x)$ of the solution $y(x)$ will be determined recurrently. Specific algorithms were seen in ([20], [21]) to formulate Adomian polynomials. The following algorithm:

$$\begin{gathered} A_0 = F(u_0), \\ A_1 = F^{'}(u_0)u_1, \\ A_2 = F^{'}(u_0)u_2 + \frac{1}{2}F^{''}(u_0)u_1^2, \\ A_3 = F^{'}(u_0)u_3 + F^{''}(u_0)u_1u_2 + \frac{1}{3!}F^{'''}(u_0)u_1^3, \\ . \\ . \\ . \end{gathered} \tag{2.7}$$

can be used to construct Adomian polynomials, when F(u) is a nonlinear function. By substituting equation (2.5) and (2.4) into gives,

$$\sum_{n=0}^{\infty} y_n = \phi(x) + L^{-1}(g(x)) + L^{-1}\sum_{n=0}^{\infty} A_n. \tag{2.8}$$

Through using Adomian decomposition method, the components $y_n(x)$ can be determined as

$$\begin{gathered} y_0 = \phi(x) + L^{-1}g(x), \\ y_{n+1} = L^{-1}A_n, n \geq 0, \end{gathered} \tag{2.9}$$

which gives

$$\begin{aligned} y_0 &= \phi(x) + L^{-1}g(x), \\ y_1 &= L^{-1}A_0, \\ y_2 &= L^{-1}A_1, \\ y_3 &= L^{-1}A_2, \\ &. \\ &. \end{aligned} \tag{2.10}$$

.

From equation (2.7) and (2.10), we can determine the components $y_n(x)$, and hence the series solution of $y(x)$ in equation (2.5) can be immediately obtained. For numerical purposes, the n-term approximant

$$\Phi_n = \sum_{n=0}^{n-1} y_k, \tag{2.11}$$

can be used to approximate the exact solution. The approach presented above can be validated by testing it on a variety of several linear and nonlinear differential equations with Dirichlet conditions.

3 Applications of MADM

In order to assess both the applicability and the accuracy of MADM, we apply MADM to several singular Lane-Emden equations as indicated in the following examples. We shall consider both linear and nonlinear problems separately.
Problem 1. Consider the following linear, homogeneous Lane-Emden equation:

$$y^{''} + \frac{2}{x}y^{'} + ky = 0, \tag{3.1}$$

$$y(b) = \alpha_2, \quad y(0) = \alpha_1.$$

According to equation (2.1), we applying L^{-1} on both sides of equation (3.1) we find

$$y(x) = \frac{\cos\sqrt{k}x}{x}\left(\frac{b\alpha_2}{\cos\sqrt{k}b} - \frac{\alpha_1}{\sqrt{k}}\tan\sqrt{k}b\right) + \frac{\alpha_1 \sin\sqrt{k}x}{\sqrt{k}x},$$

for $k = 1$, $\alpha_1 = b = 1$, $\alpha_2 = \sin 1.$, we get

$$y(x) = \frac{\sin x}{x}.$$

So, the exact solution is easily obtained by this method.
However solving problem (1) using Adomian Decomposition Method (ADM). where

$$L^{-1} = \int_0^x \int_0^x x^{-2}(.)dxdx, \tag{3.2}$$

Equation (3.1) can be expressed

$$(x^2 y^{'})^{'} = -kx^2 y, \tag{3.3}$$

Applying equation (3.2) on both sides of (3.3), for $k = 1$ and using the boundary condition at $x = 0$, yields

$$y(x) = 1 - \frac{1}{6}x + \frac{1}{120}x^4 - \frac{1}{5040}x^6 \ldots, \tag{3.4}$$

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Table 1: Comparison of numerical errors

x	EXACT	MADM	SADM	ERROR^a[MADM]	ERROR^b[SADM]
1.00	0.8414709800	0.8414709800	0.84147101	0.0000000	0.00000003
1.10	0.8101885100	0.8101885100	0.81018857	0.0000000	0.00000006
1.20	0.7766992400	0.7766992400	0.77669939	0.0000000	0.00000015
1.30	0.7411986000	0.7411986000	0.74119895	0.0000000	0.00000035
1.40	0.7038926600	0.7038926600	0.70389338	0.0000000	0.00000072
1.50	0.6649966600	0.6649966600	0.66499808	0.0000000	0.00000142
1.60	0.6247335000	0.6247335000	0.62473621	0.0000000	0.00000271
1.70	0.5833322400	0.5833322400	0.58333720	0.0000000	0.00000496
1.80	0.5410264600	0.5410264600	0.54103522	0.0000000	0.00000876
1.90	0.4980526800	0.4980526800	0.49806769	0.0000000	0.00001501
2.00	0.4546487100	0.4546487100	0.45467372	0.0000000	0.00002501

Table 1 exhibits a comparison between the errors obtained by using the proposed Modify Adomian Decomposition Method (MADM) and the Standard Adomian Decomposition Method (SADM). Examining this table closely shows the improvements obtained by using the proposed scheme. Higher accuracy can be obtained by evaluating more components of $y(x)$

Problem 2. Consider the singular boundary value problems:

$$y^{''} + \frac{2}{x}y^{'} + y = 6 + 12x + x^2 + x^3, \tag{3.5}$$

$$y(0) = 0, y(1) = 2.$$

According to equation (2.1), applying L^{-1} on both sides of equation (3.5) yields

$$y = \frac{\cos x}{x}\frac{2}{\cos 1} + L^{-1}(6 + 12x + x^2 + x^3) = x^2 + x^3.$$

Also, solving problem (2)using Adomian Decomposition Method (ADM). where

$$L^{-1} = \int_0^x \int_0^x x^{-2}(.)dxdx, \tag{3.6}$$

Equation (3.5) can be written as

$$(x^2y^{'})^{'} = 6x^2 + 12x^3 + x^4 + x^5 - x^2y \tag{3.7}$$

Applying equation (3.6) on both sides of eqaution (3.7) and using the boundary conditions at $x = 0$, yields

$$y(x) = x^2 + x^3 - \frac{1}{19958400}x^{11} - \frac{1}{6652800}x^{10}, \tag{3.8}$$

Table 2 exhibits a comparison between the errors obtained by using the proposed Modify Adomian Decomposition Method (MADM) and the Standard Adomian Decomposition Method (SADM). Examining this table closely shows the improvements obtained by using the proposed scheme. Higher accuracy can be obtained by evaluating more components of $y(x)$

Problem 3. Consider the singular boundary value problem[3,30-32]:

$$y^{''} + \frac{2}{x}y^{'} - 4y = -2, \tag{3.9}$$

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Table 2: Comparison of numerical errors

x	EXACT	MADM	SADM	ERROR^a[MADM]	ERROR^b[SADM]
0.00	0.0000000000	0.0000000000	0.00000000	0.0000000	0.00000000
0.10	0.0110000000	0.0110000000	0.01100000	0.0000000	0.00000000
0.20	0.0480000000	0.0480000000	0.04800000	0.0000000	0.00000000
0.30	0.1170000000	0.1170000000	0.11700000	0.0000000	0.00000000
0.40	0.2240000000	0.2240000000	0.22400000	0.0000000	0.00000000
0.50	0.3750000000	0.3750000000	0.37500000	0.0000000	0.00000000
0.60	0.5760000000	0.5760000000	0.57600000	0.0000000	0.00000000
0.70	0.8330000000	0.8330000000	0.83299999	0.0000000	0.00000001
0.80	1.1520000000	1.1520000000	1.15199998	0.0000000	0.00000002
0.90	1.5390000000	1.5390000000	1.53899993	0.0000000	0.00000007
1.00	2.0000000000	2.0000000000	1.99999980	0.0000000	0.00000020

$$\lim_{x\to 0} y(x) = 3.25721, y(1) = 5.5,$$

The true solution is $y(x) = 0.5 + \frac{1.3786 \sinh 2x}{x}$.
According to eqaution (2.1), we applying L^{-1} on both sides of equation (3.9) we find

$$y = \frac{-0.108106 \cosh 2x}{x} + \frac{1.6286 \sinh 2x}{x} + L^{-1}(-2)$$
$$= 0.5 - 1.5305565968614277 \cdot 10^{-16} \frac{\cosh 2x}{x} + \frac{1.3786 \sinh 2x}{x}.$$

solving problem (3) using Adomian Decomposition Method (ADM). where

$$L^{-1} = \int_0^x \int_0^x x^{-2}(.)dxdx, \tag{3.10}$$

Thus equation (3.10) can be written as

$$(x^2 y^{'})^{'} = 4x^2 y - 2x^2 \tag{3.11}$$

Applying equation (3.10) on both sides of equation (3.11) and using the boundary conditions at $x = 0$, yields

$$\begin{aligned} y(x) &= 3.25721 + 1.838140001x^2 + 0.3676280001x^4 + 0.03501219048x^6 \\ &+0.001945121694x^8 - 0.0001282667949x^{10}, \end{aligned} \tag{3.12}$$

Table 3 exhibits a comparison between the errors obtained by using the proposed Modify Adomian Decomposition Method (MADM) and the Standard Adomian Decomposition Method (SADM). Examining this table closely shows the improvements obtained by using the proposed scheme. Higher accuracy can be obtained by evaluating more components of $y(x)$

Problem 4. Consider the singular boundary value problem

$$y^{''} + \frac{2}{x} y^{'} + y = 6 + x^2 - e^{x^2} + e^y, \tag{3.13}$$

$$y(0) = 0, y(1) = 1.$$

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Table 3: Comparison of numerical errors

x	EXACT	MADM	SADM	ERROR^a[MADM]	ERROR^b[SADM]
0.10	3.2756181300	3.2756181300	3.27562820	0.0000000	0.00001007
0.20	3.3313157800	3.3313157800	3.33132605	0.0000000	0.00001027
0.30	3.4256354300	3.4256354300	3.42564604	0.0000000	0.00001061
0.40	3.5608572700	3.5608572700	3.56086836	0.0000000	0.00001109
0.50	3.7402647300	3.7402647300	3.74027640	0.0000000	0.00001167
0.60	3.9682390400	3.9682390400	3.96825111	0.0000000	0.00001207
0.70	4.2503857800	4.2503857800	4.25039700	0.0000000	0.00001122
0.80	4.5936974800	4.5936974800	4.59370323	0.0000000	0.00000575
0.90	5.0067571900	5.0067571900	5.00674388	0.0000000	0.00001331
1.00	5.4999897600	5.4999897600	5.49992249	0.0000000	0.00006727

the exact solution $y = x^2$.
By similar approach of equation (2.1), applying L^{-1} on both sides of(26) gives

$$y(x) = x^2 + \frac{1}{30}x^4 - \frac{1}{1260}x^6 \tag{3.14}$$

In a similar fashion, solving problem (4) using Adomian Decomposition Method (ADM). where

$$L^{-1} = \int_0^x \int_0^x x^{-2}(.)dxdx, \tag{3.15}$$

Equation (3.13) becomes

$$(x^2y^{'})^{'} = 6x^2 + x^4 - x^2e^{x^2} + x^2e^y - x^2y \tag{3.16}$$

Applying equation (3.15) on both sides of equation (3.16) and using the boundary conditions at $x = 0$, yields

$$y(x) = 0.99999999x^2 - \frac{1}{12}x^3 + 0.25000000001x^4 - \frac{1}{180}x^5 + 0.007275132275x^6 - 0.001240079365x^7, \tag{3.17}$$

Table 4 exhibits a comparison between the errors obtained by using the proposed Modify Adomian Decomposition Method (MADM) and the Standard Adomian Decomposition Method (SADM). Examining this table closely shows the improvements obtained by using the proposed scheme. Higher accuracy can be obtained by evaluating more components of $y(x)$

4 Discussion and Conclusion

A modified Adomian decomposition method based on a new operator has been employed successfully for the numerical solution of singular differential equation with Dirichlet boundary conditions. This method is suitable, straightforward, without restrictive assumptions, and the components of the series solution can be easily computed using any mathematical symbolic package. Moreover, this method does not change the problem into a convenient one for the use of linear terms. It therefore, provide more realistics series solution that converges rapidly to the exact solution. Numerical results show that the proposed schemes is effective, convenient and reliable for the class of problem considered.

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Table 4: Comparison of numerical errors

x	EXACT	MADM	SADM	ERROR^a[MADM]	ERROR^b[SADM]
0.0	0.000000	0.000000	0.000000	0	0
0.1	0.010000	0.010003	0.009919	3.33E-06	8.088E-05
0.2	0.040000	0.040053	0.039372	5.328E-05	0.00062799
0.3	0.090000	0.090269	0.087944	0.00026942	0.00205597
0.4	0.160000	0.160850	0.155278	0.00085008	0.00472246
0.5	0.250000	0.252071	0.241076	0.00207093	0.00892379
0.6	0.360000	0.364283	0.345113	0.00428297	0.01488729
0.7	0.490000	0.497910	0.467239	0.00790996	0.02276077
0.8	0.640000	0.653445	0.607400	0.01344528	0.03260004
0.9	0.810000	0.831448	0.765645	0.02144822	0.04435482
1.0	1.000000	1.032540	0.942146	0.03253968	0.05785384

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Conflict of interest

The authors declare that they have no conflicts of interest.

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