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Viscosity-Inertia Approximation Methods for Attractive Point of Finite Family Generalized Non Expansive Mapping

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Abstract

In this research paper, an attractive point problem involving generalized non-expansive mapping is studied, using viscosity approximation method with inertia parameters. We established strong convergence theorem for an attractive point of finite families of generalized non-expansive mapping in a uniform convex Banach space, which is also a solution of some variational inequality problems in a Banach space. Finally, we give a numerical experiment to validate the performance of our algorithm. Our results improve and extend some recent results in the literature review.

Keywords: Attractive Point, Generalized nonexpansive mapping, Inertia, Uniformly convex Banach space, Viscosity approximation method.
MSC2010: 47H09, 47J25, 47H10, 54H25.

1 Introduction

Let X be a uniformly convex Banach space and C a nonempty and convex subset of X. Let T be a mapping from C into itself, the set of fixed point of T is denoted by $F(T)= \{x \in C : Tx = x\}$. A mapping $T : C \to C$ is called

1. non-expansive mapping if $\|Tx - Ty\| \leq \|x - y\|$
2. quasi nonexpansive mapping if $F(T) \neq \emptyset$ and $\|Tx - z\| \leq \|x - z\| \ \forall \ x \in C$ and $\forall \ z \in F(T)$ [1].
3. generalized nonexpansive mapping if $\|Tx - Ty\| \leq a\|x - y\| + b\|Tx - x\| + c\|Tx - y\| + d\|Ty - y\| + e\|Ty - x\| \ \forall \ x, y \in C$ and a, b, c, d, e are non-negative constant and $a + b + c + d + e \leq 1$ [2].

It was also proved that the above definition (3) is equivalent with $\|Tx - Ty\| \leq a\|x-y\| + b\left(\|Tx-x\| + \|Ty-y\|\right) + c\left(\|Tx-y\| + \|Ty-x\|\right) \; \forall \; x, y \in C$, where a, b and c are nonnegative constants such that $a + 2b + 2c \leq 1$. By letting $a = 1$ and $b = c = 0$, we can see that every nonexpansive mapping is a generalized nonexpansive mapping. It is well-known from [3] that every generalized nonexpansive mapping with a fixed point is a quasi-nonexpansive mapping.

Many Researchers have been working on attractive point in various directions. The concept of attractive points was first introduced in Hilbert space in [4]. The introduction was basically motivated to get rid of the closedness and convexity hypotheses imposed on nonempty subset C of a Hilbert space (H) in a celebrated Bailons [5] nonlinear ergodic theorem. They established nonlinear ergodic theorem without convexity for generalized hybrid mappings in Hilbert spaces. They also proved an existence theorem for attractive points of some nonlinear mappings without assuming convexity of its domain. From the definition of sets of attractive points and fixed points $A\left(T\right) = \{u \in H : \|Tx - u\| x \in C\}$ and $F(T) = \{u \in C : Tu = u\}$, where C is a nonempty closed convex subset of a real Hilbert space H and $T : C \to H$ is a nonlinear mapping, it is easily seen that neither an attractive point is a fixed point nor conversely. However, [4] gave the relation between the two in the following lemmas.

Lemma 1.1. *Let H be a real Hilbert space, and let C be a non-empty closed convex subset of H. Let $T : C \to C$ be a mapping. If $A\left(T\right) \neq \emptyset$, then $F\left(T\right) \neq \emptyset$.*

Lemma 1.2. *Let H be a real Hilbert space, and let C be a nonempty subset of H. Let $T : C \to H$ be a quasi-nonexpansive mapping $\|Tx - z\| \leq \|x - z\|, z \in F\left(T\right), x \in C$. Then $A\left(T\right) \cup C = F(T)$.*

The concept of attractive points for nonlinear mapping in Banach spaces was first introduced in [6]. Subsequently, in [7] they proved convergence theorems for attractive points of some generalized non-expansive mappings in uniformly convex Banach spaces. Since then, the theory of attractive points has attracted considerable attention, and various authors have investigated the concept using different analytical techniques: see, for example [8], [9], [10]. In 2018, [11] further extended the concept of attractive points to the setting of two mappings in Hilbert spaces. Let $T_1, T_2 : C \to H$, where C is a nonempty closed convex subset of H. The set of all common attractive points for T_1 and T_2 is denoted by $A\left(T_1, T_2\right)$ and defined as
$A\left(T_1, T_2\right) = \{u \in H : \max\left(\|T_1x - u\|, \|T_2x - u\|\right) \leq \|x - u\|, \forall x \in C\}$.
$A\left(T_1, T_2\right) = A\left(T_1\right) \cap A\left(T_2\right)$ and the common attractive point for finite family of nonlinear mappings is denoted by $A\left(T_i\right)$, for each $i = 1, 2, 3, \ldots, n$
$A\left(T_i\right) = \left\{ u \in H : \max\limits_{1 \leq i \leq n} \left(\|T_ix - u\|\right) \leq \|x - u\|, \forall x \in C \right\}$.

The authors [12] proved a convergence theorem for common fixed points of the Mann's iteration for two generalized non-expansive mappings in uniformly convex Banach spaces.

$$\begin{cases} x_1 \in C \\ x_{n+1} = \alpha_n x_n + \beta_n T_1 x_n + \gamma_n T_2 x_n \qquad \forall n \in N \end{cases} \tag{1.1}$$

α_n, β_n, $\gamma_n \in (0, 1)$ and $\alpha_n + \beta_n + \gamma_n = 1$.
The viscosity approximation was introduced by [13], which is defined as follows: For : $x_1 \in C$
$x_{n+1} = \alpha_n \, f\left(x_n\right) + \left(1 - \alpha_n\right) Tx_n \qquad \forall$ n $\in N$ where $\alpha_n \in [0, 1]$ and f is a contraction mapping under some suitable control conditions and proved that x_n converges strongly to a fixed point of T, where T is a nonexpansive mapping. The authors, in [14] introduced a new accelerated algorithm for finding a common fixed point of an infinite family of non-expansive mappings T_i in Hilbert space based on the concept of Inertia forward and backward of Mann's viscosity algorithm.

Most recently, [15] they proved strong convergence theorems for common attractive points of two generalized nonexpansive mappings without assuming the closedness of the domain. Using iterative (1.1) in uniformly convex Banach space. They obtained strong convergence by imposing compactness assumption on the domain and using the so called condition A, that is, there exists a

non-negative non-decreasing function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$ and $h\,(0) > 0$ for any $r > 0$ such that $h\left(d\left(x_n, A\left(T_1, T_2\right)\right)\right) \leq \|x_n - T_1x_n\|$ or $h\left(d\left(x_n, A\left(T_1, T_2\right)\right)\right) \leq \|x_n - T_2x_n\|$.
Inspired and motivated by the above results, it is our purpose in this paper to extend and generalize the result in [15] from two generalized nonexpansive mappings to finite family and furthermore, we introduce a viscosity and inertia parameter to prove strong convergence theorem to common attractive points of the said mappings.

2 Preliminaries

Definition 2.1. *A Banach space X is called uniformly convex Banach space if for any $\epsilon \in (0, 2]$ there exist $\delta = \delta(\epsilon)$ such that for all $x, y \in X$ with $\|x\|, \|y\| \leq 1$ and*
$\|x - y\| \geq \epsilon$ then $\|\frac{x+y}{2}\| \leq$1-δ.

Definition 2.2.

1. *Demiclosed at $y_o \in C$, if for any sequence $\{x_n\}$ in C which converges weakly to $x_o \in C$ and $Tx_n \to y_o$, it holds that $Tx_o = y_o$.*
2. *Semicompact, if for any bounded sequence $\{x_n\}$ in C such that $\lim\limits_{n\to\infty} ||x_n - Tx_n|| = 0$. There exist a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x^* \in C$ as $k \to \infty$.*

The following lemmas will be needed in the proof of the main results.

Lemma 2.3. *[16] Suppose T: $C \to C$ is a nonexpansive mapping. Let (x_n) be a sequence in X and u be a fixed element in X. If $x_n \rightharpoonup u$ and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$ then $u \in A\,(T_i)$.*

Lemma 2.4. *[17] Let X be a uniformly convex Banach space. For arbitrary $r > 0$,*
let $B_r(0) = \{x \in X : \|x\| \leq r\}$. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that
$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2$- $\lambda\ \mu\ g(\|x - y\|)$ $\quad \forall x, y \in B_r(0)$ and $\lambda,\ \mu,\ \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$

Lemma 2.5. *[18] Let X be a uniformly convex Banach space and $B_r(0)$ with arbitrary $r > 0$ be a closed ball of X. Then for any given sequence $\{x_n\} \subset B_r(0)$ and for any $\lambda_j \in [0, 1]\,, j = 1, 2, \ldots s$ with $\sum\limits_{j=1}^{S} \lambda_j = 1$ there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that for any $j, i, k \in \{1, 2, \ldots S\}$ with $j < i, k$ the following inequality holds $\|\sum\limits_{j=1}^{S} \lambda_j x_j\|^2 \leq \sum\limits_{j=1}^{S} \|x_j\|^2$ - $\lambda_i\lambda_k g\left(\|x_i - x_k\|\right)$.*

Lemma 2.6. *[19] Let $\{S_n\}$ and $\{b_n\}$ be sequences of non-negative real sequence, $\{\sigma_n\}$ a sequence in $[0, 1]$ and $\{t_n\}$ be a sequence of real numbers such that*
$S_{n+1} \leq (1 - \sigma_n)\ S_n\ + \sigma_n t_n + b_n$. If the following hold

1. $\sum\limits_{n=1}^{\infty} \sigma_n = \infty$
2. $\sum\limits_{n=1}^{\infty} b_n < \infty$,
3. $\limsup\limits_{n\to\infty} t_n \leq 0$ *then* $\lim\limits_{n\to\infty} S_n = 0$.

Lemma 2.7. *[20] Let $\{\Phi_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Phi_{n_t}\}$ of $\{\Phi_n\}$ which satisfies $\Phi_{n_t} < \Phi_{n_{t+1}}$ for all $i \in \mathbb{N}$. Let $\{\tau(n)\}_{n \geq n_0}$ be a sequence of integer, defined as follows:*

$$\tau(n) := \max\{s \leq n : \Phi_s < \Phi_{s+1}\},$$

where $n_0 \in \mathbb{N}$ *such that* $\{s \leq n_0 : \Phi_s < \Phi_{s+1}\} \neq \emptyset$. *Then the followings are satisfied:*

1. $\tau(n_0) \leq \tau(n_0+1) \leq \dots$ *and* $\tau(n) \to \infty$,
2. $\Phi_{\tau(n)} \leq \Phi_{\tau(n)+1}$ *and* $\Phi_n \leq \Phi_{\tau(n)+1}$, *for all* $n \geq n_0$.

.

3 Main Results

In this section, we prove strong convergence theorems for finite family generalized nonexpansive mapping in real uniformly convex Banach space.

Theorem 3.1. *Let* X *be a uniformly Convex Banach space. Suppose* $\{T_i\}$ *is a finite family of generalized nonexpansive mapping and* $T_i : X \to X$ *for each* $i = 1, 2, \dots, N$ *be a finite family of generalized nonexpensive mapping with* $\bigcap_{i=1}^{N} A(T_i) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence defined by*

$$\begin{cases} x_0, x_1 \in X \\ z_n = x_n + \theta_n (x_n - x_{n-1}) \qquad n \geq 1 \\ y_n = \sum_{i=1}^{N+1} \gamma_{n,i} T_{i-1} z_n \qquad \forall \quad i = 1, 2, 3, ..., N+1 \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \Omega_n y_n \qquad \forall \quad n \geq 1 \end{cases} \tag{3.1}$$

where $f : X \to X$ *is a contraction mapping with* $K \in [0, 1)$, T_o *is identity mapping* $(T_o = I)$ *with* (α_n), (β_n), (Ω_n), $(\gamma_{n,i})$ *are sequences in* $(0, 1)$. *Then sequence* $\{x_n\}$ *converges strongly to the common attractive points* $u \in \bigcap_{i=1}^{N} A(T_i)$ *provided that the following conditions holds:*

1. $\alpha_n + \beta_n + \Omega_n = 1 \qquad \forall\ n \geq 1$
2. $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$
3. $\lim_{n\to\infty} \theta_n \|x_n - x_{n-1}\| = 0$ *and* $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0$
4. $\sum_{i=1}^{N+1} \gamma_{n,i} = 1$
5. $0 < a \leq \beta_n \leq b < 1$
6. $0 < c \leq \Omega_n \leq d < 1$.

Proof. We divide the proof into the following lemmas:

Lemma 3.2. *Let* X *be a Uniformly convex Banach space. Suppose* $\{T_i\}$ *is a finite family of generalized nonexpansive mapping and* $T_i : X \to X$ *for each* $i = 1, 2, 3, \dots, N$ *with* $\bigcap_{i=1}^{N} A(T_i) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by* (3.1), *this implies* $\{x_n\}$ *is bounded.*

Proof:

Let $u \in \bigcap_{i=1}^{N} A(T_i)$. From the scheme (3.1) above, we have:

$$\|z_n - u\| = \|x_n + \theta_n (x_n - x_{n-1}) - u\|.$$

Hence

$$\|z_n - u\| \leq \|x_n - u\| + \theta_n \|x_n - x_{n-1}\|. \tag{3.2}$$

Also

$$\begin{aligned}
\|y_n - u\| &= \| \sum_{i=1}^{N+1} \gamma_{n,i} T_{i-1} z_n - u\| \\
&\leq \sum_{i=1}^{N+1} \|\gamma_{n,i} \left(T_{i-1} z_n - u \right) \| \\
&\leq \sum_{i=1}^{N+1} \gamma_{n,i} \|z_n - u\| \\
\|y_n - u\| &\leq \|z_n - u\|.
\end{aligned}$$

Using (3.2), we have

$$\|y_n - u\| \leq \|x_n - u\| + \theta_n \|x_n - x_{n-1}\|. \tag{3.3}$$

Now from (3.2), (3.3), and scheme (3.1), we have

$$\begin{aligned}
&\|x_{n+1} - u\| = \|\alpha_n f\left(x_n\right) + \beta_n x_n + \Omega_n y_n - u\| \\
&= \|\alpha_n \left(f\left(x_n\right) - u\right) + \beta_n \left(x_n - u\right) + \Omega_n \left(y_n - u\right) \| \\
&\leq \alpha_n \|f\left(x_n\right) - u\| + \beta_n \|x_n - u\| + \Omega_n \|y_n - u\| \\
&= \alpha_n \|f\left(x_n\right) - f\left(u\right) + f\left(u\right) - u\| + \beta_n \|x_n - u\| + \Omega_n \|y_n - u\| \\
&\leq \alpha_n \|f\left(x_n\right) - f\left(u\right) \| + \alpha_n \|f\left(u\right) - u\| + \beta_n \|x_n - u\| + \Omega_n \|y_n - u\| \\
&\leq \alpha_n K \|x_n - u\| + \beta_n \|x_n - u\| + \Omega_n \left(\|x_n - u\| + \theta_n \|x_n - x_{n-1}\|\right) + \alpha_n \|f\left(u\right) - u\| \\
&= \alpha_n K \|x_n - u\| + \left(\beta_n + \Omega_n\right) \|x_n - u\| + \Omega_n \theta_n \|x_n - x_{n-1}\| + \alpha_n \|f\left(u\right) - u\| \\
&= \alpha_n K \|x_n - u\| + \left(1 - \alpha_n\right) \|x_n - u\| + \Omega_n \theta_n \|x_n - x_{n-1}\| + \alpha_n \|f\left(u\right) - u\| \\
&= \left(\alpha_n K + 1 - \alpha_n\right) \|x_n - u\| + \Omega_n \theta_n \|x_n - x_{n-1}\| + \alpha_n \|f\left(u\right) - u\| \\
&= \Big(1 - \alpha_n \left(1 - K\right)\Big) \|x_n - u\| + \Omega_n \theta_n \|x_n - x_{n-1}\| + \alpha_n \|f\left(u\right) - u\| \\
&= \Big(1 - \alpha_n \left(1 - K\right)\Big) \|x_n - u\| + \frac{\alpha_n \left(1 - K\right)}{\alpha_n \left(1 - K\right)} \Big[\Omega_n \theta_n \|x_n - x_{n-1}\| + \alpha_n \|f\left(u\right) - u\|\Big] \\
&= \Big(1 - \alpha_n \left(1 - K\right)\Big) \|x_n - u\| + \alpha_n \left(1 - K\right) \Big[\frac{\Omega_n}{1 - K} \times \frac{\theta_n}{\alpha_n} \times \|x_n - x_{n-1}\| \\
&\quad + \frac{\alpha_n}{\alpha_n \left(1 - K\right)} \times \|f\left(u\right) - u\|\Big] \\
&\leq \Big(1 - \alpha_n \left(1 - K\right)\Big) \|x_n - u\| + \alpha_n \left(1 - K\right) \Big[\frac{d}{1 - K} \times \frac{\theta_n}{\alpha_n} \times \|x_n - x_{n-1}\| + \frac{\|f\left(u\right) - u\|}{1 - K}\Big].
\end{aligned}$$

By condition (3), we have $\frac{\theta_n}{\alpha_n} \ \|x_n - x_{n-1}\| \to 0$ as $n \to \infty$ that is $\exists$ a positive Constant

$M_1 > 0$ such that $\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \le M_1 \ \forall\ n \ge 1$.

$$\begin{aligned}
\|x_{n+1}-u\| &\le \Big(1-\alpha_n(1-K)\Big)\|x_n-u\| + \alpha_n(1-K)\left[\frac{dM_1}{1-K} + \frac{\|f(u)-u\|}{1-K}\right] \\
&= \Big(1-\alpha_n(1-K)\Big)\|x_n-u\| + \alpha_n(1-K)\left[\frac{dM_1+\|f(u)-u\|}{1-K}\right] \\
&\le \Big(1-\alpha_n(1-K)\Big)\max\left\{\|x_n-u\|, \frac{dM_1+\|f(u)-u\|}{1-K}\right\} \\
&\quad + \alpha_n(1-K)\max\left\{\|x_n-u\|, \frac{dM_1+\|f(u)-u\|}{1-K}\right\} \\
&= \max\left\{\|x_n-u\|, \frac{dM_1+\|f(u)-u\|}{1-K}\right\} \\
&\quad - \alpha_n(1-K)\max\left\{\|x_n-u\|, \frac{dM_1+\|f(u)-u\|}{1-K}\right\} \\
&\quad + \alpha_n(1-K)\max\left\{\|x_n-u\|, \frac{dM_1+\|f(u)-u\|}{1-K}\right\} \\
\|x_{n+1}-u\| &\le \max\left\{\|x_n-u\|, \frac{dM_1+\|f(u)-u\|}{1-K}\right\} \\
&\ \vdots \\
&\le \max\left\{\|x_0-u\|, \frac{dM_1+\|f(u)-u\|}{1-K}\right\}. \qquad (3.4)
\end{aligned}$$

Therefore $\{x_n\}$ is bounded. It follows that $\{z_n\}$, $\{y_n\}$, $\{f(x_n)\}$, $\{T_i x_n\}$ $\{T_i z_n\}$ are all bounded.

Lemma 3.3. *Let $\{x_n\}$ be a sequence in Lemma 3.2, (α_n), (β_n), (Ω_n), $(\gamma_{n,i})$ are sequences in $(0,1)$ that satisfied the conditions in (3.1), then $\{x_n\}$ converges strongly to a point $u \in \bigcap_{i=1}^{N} A(T_i)$ which solves the variational inequality*

$$\langle f(u)-u, j(u-z)\rangle \ge 0 \quad \forall z \in \bigcap_{i=1}^{N} A(T_i). \qquad (3.5)$$

Proof. By the definition of z_n and y_n, we have

$$\begin{aligned}
\|z_n-u\|^2 &= \|x_n+\theta_n(x_n-x_{n-1})-u\|^2 \\
&\le \Big(\ \|x_n-u\| + \theta_n\|x_n-x_{n-1}\|\Big)^2.
\end{aligned}$$

Hence, we have

$$\|z_n-u\|^2 \quad \le \|x_n-u\|^2 + 2\theta_n\|x_n-x_{n-1}\|\|x_n-u\| + \theta_n^2\|x_n-x_{n-1}\|^2. \qquad (3.6)$$

Also, we have

$$\begin{aligned}
\|y_n-u\|^2 &= \|\sum_{i=1}^{N+1}\gamma_{n,i}T_{i-1}z_n - u\|^2 \\
&= \|\sum_{i=1}^{N+1}\gamma_{n,i}\left(T_{i-1}z_n-u\right)\|^2 \\
&\le \left(\sum_{i=1}^{N+1}\gamma_{n,i}\|T_{i-1}z_n-u\|\right)^2 \\
&\le \Big(\|x_n-u\| + \theta_n\|x_n-x_{n-1}\|\Big)^2.
\end{aligned}$$

Therefore,

$$\|y_n-u\|^2 \le \|x_n-u\|^2 + 2\theta_n\|x_n-x_{n-1}\|\|x_n-u\| + \theta_n^2\|x_n-x_{n-1}\|^2. \tag{3.7}$$

From (3.3), (3.7), and scheme (3.1), we have

$$\begin{aligned}
&\|x_{n+1}-u\|^2 \le \|\alpha_n f(x_n) + \beta_n x_n + \Omega_n y_n - u\|^2\\
&= \|\alpha_n (f(x_n)-u) + \beta_n (x_n-u) + \Omega_n (y_n-u)\|^2\\
&= \|\alpha_n (f(x_n)-f(u)+f(u)-u) + \beta_n (x_n-u) + \Omega_n (y_n-u)\|^2\\
&= \|\alpha_n \big(f(x_n)-f(u)\big) + \alpha_n\big(f(u)-u\big) + \beta_n (x_n-u) + \Omega_n (y_n-u)\|^2\\
&\le \|\alpha_n \big(f(x_n)-f(u)\big) + \beta_n (x_n-u) + \Omega_n (y_n-u)\|^2 + 2\langle \alpha_n\big(f(u)-u\big), j(x_{n+1}-u)\rangle\\
&\le \alpha_n\|f(x_n)-f(u)\|^2 + \beta_n\|x_n-u\|^2 + \Omega_n\big[\|x_n-u\|^2 + 2\theta_n\|x_n-x_{n-1}\|\|x_n-u\|\\
&\quad + \theta_n^2\|x_n-x_{n-1}\|^2\big] + 2\alpha_n\langle\big(f(u)-u\big), j(x_{n+1}-u)\rangle.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&\|x_{n+1}-u\|^2 = \alpha_n\|f(x_n)-f(u)\|^2 + \beta_n\|x_n-u\|^2 + \Omega_n\|x_n-u\|^2\\
&\quad + 2\Omega_n\theta_n\|x_n-x_{n-1}\|\|x_n-u\| + \Omega_n\theta_n^2\|x_n-x_{n-1}\|^2 + 2\alpha_n\langle\big(f(u)-u\big), j(x_{n+1}-u)\rangle\\
&= \alpha_n\|f(x_n)-f(u)\|^2 + (\beta_n+\Omega_n)\|x_n-u\|^2\\
&\quad + \Omega_n\theta_n\|x_n-x_{n-1}\|\big[2\|x_n-u\| + \theta_n\|x_n-x_{n-1}\|\big] + 2\alpha_n\langle\big(f(u)-u\big), j(x_{n+1}-u)\rangle\\
&= \alpha_n\|f(x_n)-f(u)\|^2 + (1-\alpha_n)\|x_n-u\|^2 + \Omega_n\theta_n\|x_n-x_{n-1}\|\\
&\quad \big[2\|x_n-u\| + \theta_n\|x_n-x_{n-1}\|\big] + 2\alpha_n\langle\big(f(u)-u\big), j(x_{n+1}-u)\rangle\\
&\le \alpha_n K\|x_n-u\|^2 + (1-\alpha_n)\|x_n-u\|^2 + \Omega_n\theta_n\|x_n-x_{n-1}\|\big[2\|x_n-u\| + \theta_n\|x_n-x_{n-1}\|\big]\\
&\quad + 2\alpha_n\langle\big(f(u)-u\big), j(x_{n+1}-u)\rangle\\
&= (\alpha_n K + 1 - \alpha_n)\|x_n-u\|^2 + \Omega_n\theta_n\|x_n-x_{n-1}\|\big[2\|x_n-u\| + \theta_n\|x_n-x_{n-1}\|\big]\\
&\quad + 2\alpha_n\langle\big(f(u)-u\big), j(x_{n+1}-u)\rangle.
\end{aligned}$$

By condition (3) we have $\theta_n\ \|x_n-x_{n-1}\| \to 0$ as $n\to\infty$, there exists a positive constant $M_2>0$ such that $\theta_n\ \|x_n-x_{n-1}\| \le M_2 \quad \forall\ n\ge 1$.

$$\begin{aligned}
\|x_{n+1}-u\|^2 &\le \big(1-\alpha_n(1-K)\big)\|x_n-u\|^2 + \Omega_n\theta_n\|x_n-x_{n-1}\|\big[2\|x_n-u\| + M_2\big]\\
&\quad + 2\alpha_n\langle\big(f(u)-u\big), j(x_{n+1}-u)\rangle.
\end{aligned}$$

Now $2\|x_n-u\| + M_2 \le 3\max\{\|x_n-u\|, M_2\}$ by taking $M_3 = \max\{\|x_n-u\|, M_2\}$, we have

$$\begin{aligned}
\|x_{n+1}-u\|^2 &\le \big(1-\alpha_n(1-K)\big)\|x_n-u\|^2 + \Omega_n\theta_n\|x_n-x_{n-1}\|3M_3\\
&\quad + 2\alpha_n\langle\big(f(u)-u\big), j(x_{n+1}-u)\rangle\\
&= \big(1-\alpha_n(1-K)\big)\|x_n-u\|^2 + \frac{\alpha_n(1-K)}{\alpha_n(1-K)}\Big[\Omega_n\theta_n\|x_n-x_{n-1}\|3M_3\\
&\quad + 2\alpha_n\langle\big(f(u)-u\big), j(x_{n+1}-u)\rangle\Big]\\
&= \big(1-\alpha_n(1-K)\big)\|x_n-u\|^2 + \alpha_n(1-K)\\
&\quad \left[\frac{3M_3\Omega_n}{1-K} \times \frac{\theta_n\|x_n-x_{n-1}\|}{\alpha_n} + \frac{2\alpha_n}{\alpha_n(1-K)}\langle\big(f(u)-u\big), j(x_{n+1}-u)\rangle\right]\\
&\le \big(1-\alpha_n(1-K)\big)\|x_n-u\|^2 + \alpha_n(1-K)\\
&\quad \left[\frac{3dM_3}{1-K} \times \frac{\theta_n\|x_n-x_{n-1}\|}{\alpha_n} + \frac{2}{1-K}\langle\big(f(u)-u\big), j(x_{n+1}-u)\rangle\right]\\
\|x_{n+1}-u\|^2 &\le \big(1-\alpha_n(1-K)\big)\|x_n-u\|^2 + \alpha_n(1-K)\\
&\quad \left[\frac{3dM_3}{1-K} \times \frac{\theta_n\|x_n-x_{n-1}\|}{\alpha_n} + \frac{2}{(1-K)}\langle\big(f(u)-u\big), j(x_{n+1}-u)\rangle\right].
\end{aligned}$$

Comparing with Lemma 2.6, we have

$$\begin{aligned} s_{n+1} &= \|x_{n+1}-u\|^2 \\ s_n &= \|x_n-u\|^2 \\ t_n &= \frac{3dM_3}{1-K} \times \frac{\theta_n\|x_n-x_{n-1}\|}{\alpha_n} + \frac{2}{1-K}\langle (f(u)-u), j(x_{n+1}-u)\rangle. \end{aligned}$$

Then, we obtain

$$S_{n+1} \leq (1-\sigma_n) S_n + \sigma_n t_n \forall n \geq 1.$$

We proceed in two cases.

In the first case, we suppose that there exists $n_0 \in N$ such that the sequence $\{\|x_n-u\|\}_{n\geq n_0}$ is nonincreasing. Since the sequence $\{x_n\}$ is bounded, it follows that $\{\|x_n-u\|\}$ is a convergence sequence. Using condition (2) we get $\sum\limits_{n=1}^{\infty}\sigma_n{=}\infty$.

We next claim $\limsup\limits_{n\to\infty}\langle (f(u)-u), j(x_{n+1}-u)\rangle \leq 0$.

Coming back form the definition of y_n, we have

$$\begin{aligned} \|y_n-u\|^2 &= \|\sum_{i=1}^{N+1}\gamma_{n,i}T_{i-1}z_n-u\|^2 \\ &= \left(\|\sum_{i=1}^{N+1}\gamma_{n,i}T_{i-1}z_n-u\|\right)^2. \end{aligned}$$

By Lemma 2.5, we have

$$\begin{aligned} \|y_n-u\|^2 &\leq \sum_{i=1}^{N+1}\gamma_{n,i}\|z_n-u\|^2-\gamma_{n,1}\gamma_{n,i}g\big(\|z_n-T_{i-1}z_n\|\big) \\ &= \|z_n-u\|^2-\gamma_{n,1}\gamma_{n,i}g\big(\|z_n-T_{i-1}z_n\|\big). \end{aligned}$$

We obtain

$$\|y_n-u\|^2 \quad \leq \|z_n-u\|^2-\gamma_{n,1}\gamma_{n,i}g\big(\|z_n-T_{i-1}z_n\|\big). \tag{3.8}$$

Using the scheme (3.1), we have

$$\begin{aligned} \|x_{n+1}-u\|^2 &= \|\alpha_n f(x_n)+\beta_n x_n+\Omega_n y_n-u\|^2 \\ &\leq \alpha_n\|f(x_n)-u\|^2+\beta_n\|x_n-u\|^2+\Omega_n\|y_n-u\|^2. \end{aligned}$$

Using (3.8), we have

$$\begin{aligned} \|x_{n+1}-u\|^2 &\leq \alpha_n\|f(x_n)-u\|^2+\beta_n\|x_n-u\|^2+\Omega_n\big[\|z_n-u\|^2 \\ &\qquad -\gamma_{n,1}\gamma_{n,i}g\big(\|z_n-T_{i-1}z_n\|\big)\big] \\ &\leq \alpha_n\|f(x_n)-u\|^2+\beta_n\|x_n-u\|^2+\Omega_n\big[\|x_n-u\|^2+2\theta_n\|x_n-x_{n-1}\| \\ &\quad \|x_n-u\|+\theta_n^2\|x_n-x_{n-1}\|^2\big]-\Omega_n\gamma_{n,1}\gamma_{n,i}g\big(\|z_n-T_{i-1}z_n\|\big) \\ &= \alpha_n\|f(x_n)-u\|^2+(\beta_n+\Omega_n)\|x_n-u\|^2+\Omega_n\big[2\theta_n\|x_n-x_{n-1}\|\|x_n-u\| \\ &\quad +\theta_n^2\|x_n-x_{n-1}\|^2\big]-\Omega_n\gamma_{n,1}\gamma_{n,i}g\big(\|z_n-T_{i-1}z_n\|\big) \\ &= \alpha_n\|f(x_n)-u\|^2+(1-\alpha_n)\|x_n-u\|^2+\Omega_n\theta_n\|x_n-x_{n-1}\| \\ &\quad \big[2\|x_n-u\|+\theta_n\|x_n-x_{n-1}\|\big]-\Omega_n\gamma_{n,1}\gamma_{n,i}g\big(\|z_n-T_{i-1}z_n\|\big) \\ &= \alpha_n\|f(x_n)-u\|^2+\|x_n-u\|^2-\alpha_n\|x_n-u\|^2+\Omega_n\theta_n\|x_n-x_{n-1}\| \\ &\quad \big[2\|x_n-u\|+\theta_n\|x_n-x_{n-1}\|\big]-\Omega_n\gamma_{n,1}\gamma_{n,i}g\big(\|z_n-T_{i-1}z_n\|\big) \\ \|x_{n+1}-u\|^2 &\leq \alpha_n\big(\|f(x_n)-u\|^2-\|x_n-u\|^2\big)+\|x_n-u\|^2+\Omega_n\theta_n\|x_n-x_{n-1}\| \\ &\quad \big[2\|x_n-u\|+\theta_n\|x_n-x_{n-1}\|\big]-\Omega_n\gamma_{n,1}\gamma_{n,i}g\big(\|z_n-T_{i-1}z_n\|\big). \end{aligned}$$

Consequently, we have

$$\begin{aligned}
&\Omega_n\gamma_{n,1}\gamma_{n,i}g\big(\|z_n-T_{i-1}z_n\|\big)\le\alpha_n\big(\|f(x_n)-u\|^2-\|x_n-u\|^2\big)-\|x_{n+1}-u\|^2+\|x_n-u\|^2\\
&\quad+\Omega_n\theta_n\|x_n-x_{n-1}\|\big[2\|x_n-u\|+\theta_n\|x_n-x_{n-1}\|\big]\\
&=\alpha_n\big(\|f(x_n)-u\|^2-\|x_n-u\|^2\big)-\big(\|x_{n+1}-u\|^2-\|x_n-u\|^2\big)\\
&\quad+\Omega_n\theta_n\|x_n-x_{n-1}\|\big[2\|x_n-u\|+\theta_n\|x_n-x_{n-1}\|\big].
\end{aligned}$$

Then, we have

$$\begin{aligned}
\lim_{n\to\infty}\Omega_n\gamma_{n,1}\gamma_{n,i}g\big(\|z_n-T_{i-1}z_n\|\big)\le{}&\lim_{n\to\infty}\alpha_n\big(\|f(x_n)-u\|^2-\|x_n-u\|^2\big)\\
&-\lim_{n\to\infty}\big(\|x_{n+1}-u\|^2-\|x_n-u\|^2\big)\\
&+\lim_{n\to\infty}\Omega_n\theta_n\|x_n-x_{n-1}\|\big[2\|x_n-u\|+\theta_n\|x_n-x_{n-1}\|\big].
\end{aligned}$$

It follows from the conditions (2), (6) and the convergence of the sequences $\{\|x_n-u\|\}$ and of $\{\theta_n\|x_n-x_{n-1}\|\}$, and the boundedness of $\gamma_{n,1}$ $\gamma_{n,i}$ and by virtue of the properties of the function "g", we conclude that

$$\lim_{n\to\infty}\|z_n-T_{i-1}z_n\|=0. \tag{3.9}$$

Also, we obtain

$$\begin{aligned}
\|z_n-x_n\|&=\|x_n+\theta_n(x_n-x_{n-1})-x_n\|\\
&\le\theta_n\|x_n-x_{n-1}\|\\
\lim_{n\to\infty}\|z_n-x_n\|&\le\lim_{n\to\infty}\theta_n\|x_n-x_{n-1}\|.
\end{aligned}$$

By assumptions (3), we have

$$\lim_{n\to\infty}\|z_n-x_n\|=0. \tag{3.10}$$

Again, we have

$$\begin{aligned}
\|x_{n+1}-u\|^2&=\|\alpha_nf(x_n)+\beta_nx_n+\Omega_ny_n-u\|^2\\
&=\|\alpha_n(f(x_n)-u)+\beta_n(x_n-u)+\Omega_n(y_n-u)\|^2.
\end{aligned}$$

Applying Lemma 2.4 and (3.7), we have

$$\begin{aligned}
&\|x_{n+1}-u\|^2\le\alpha_n\|f(x_n)-u\|^2+\beta_n\|x_n-u\|^2+\Omega_n\|y_n-u\|^2-\beta_n\Omega_ng(\|x_n-y_n\|)\\
&\le\alpha_n\|f(x_n)-u\|^2+\beta_n\|x_n-u\|^2+\Omega_n\big[\|x_n-u\|^2+2\theta_n\|x_n-x_{n-1}\|\|x_n-u\|\\
&\quad+\theta_n^2\|x_n-x_{n-1}\|^2\big]-\beta_n\Omega_ng\big(\|x_n-y_n\|\big)\\
&=\alpha_n\|f(x_n)-u\|^2+(1-\alpha_n)\|x_n-u\|^2+\Omega_n\theta_n\|x_n-x_{n-1}\|\\
&\quad\big[2\|x_n-u\|+\theta_n\|x_n-x_{n-1}\|\big]-\beta_n\Omega_ng\big(\|x_n-y_n\|\big).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\beta_n\Omega_ng\big(\|x_n-y_n\|\big)\le{}&\alpha_n\big(\|f(x_n)-u\|^2-\|x_n-u\|^2\big)-\big(\|x_{n+1}-u\|^2-\|x_n-u\|^2\big)\\
&+\Omega_n\theta_n\|x_n-x_{n-1}\|\big[2\|x_n-u\|+\theta_n\|x_n-x_{n-1}\|\big].
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\lim_{n\to\infty}\beta_n\Omega_ng\big(\|x_n-y_n\|\big)\le\lim_{n\to\infty}\alpha_n\big(\|f(x_n)-u\|^2-\|x_n-u\|^2\big)\\
&\quad-\lim_{n\to\infty}\big(\|x_{n+1}-u\|^2-\|x_n-u\|^2\big)\\
&\quad+\lim_{n\to\infty}\Omega_n\theta_n\|x_n-x_{n-1}\|\big[2\|x_n-u\|+\theta_n\|x_n-x_{n-1}\|\big].
\end{aligned}$$

By condition (2), (5) and (6), the convergence of the sequences $\{\|x_n - u\|\}$ and of $\{\theta_n \|x_n - x_{n-1}\|\}$, and by the property of "g", we have

$$\begin{aligned} \lim_{n\to\infty} \beta_n \Omega_n g\big(\|x_n - y_n\|\big) &\leq 0 \\ \lim_{n\to\infty} g\big(\|x_n - y_n\|\big) &= 0. \end{aligned}$$

Hence, we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.11}$$

Also, we have

$$\begin{aligned} \|y_n - z_n\| &= \|y_n - x_n + x_n - z_n\| \\ &\leq \|y_n - x_n\| + \|x_n - z_n\| \\ \lim_{n\to\infty} \|y_n - z_n\| &\leq \lim_{n\to\infty} \|y_n - x_n\| + \lim_{n\to\infty} \|x_n - z_n\|. \end{aligned}$$

By (3.10) and (3.11), we have

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{3.12}$$

Again, we have

$$\begin{aligned} \|x_{n+1} - z_n\| &= \|\alpha_n f\left(x_n\right) + \beta_n x_n + \Omega_n y_n - z_n\| \\ &= \|\alpha_n \left(f\left(x_n\right) - z_n\right) + \beta_n \left(x_n - z_n\right) + \Omega_n \left(y_n - z_n\right)\| \\ \lim_{n\to\infty} \|x_{n+1} - z_n\| &\leq \lim_{n\to\infty} \alpha_n \|f\left(x_n\right) - z_n\| + \lim_{n\to\infty} \beta_n \|x_n - z_n\| + \lim_{n\to\infty} \|y_n - z_n\|. \end{aligned}$$

By condition (1), (3.10) and (3.12), we have

$$\lim_{n\to\infty} \|x_{n+1} - z_n\| = 0. \tag{3.13}$$

Also, we compute

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|x_{n+1} - z_n + z_n - x_n\| \\ &\leq \|x_{n+1} - z_n\| + \|z_n - x_n\| \\ \lim_{n\to\infty} \|x_{n+1} - x_n\| &\leq \lim_{n\to\infty} \|x_{n+1} - z_n\| + \lim_{n\to\infty} \|z_n - x_n\|. \end{aligned}$$

Using (3.10) and (3.13), we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.14}$$

We suppose

$$b = \langle \left(f\left(u\right) - u\right), j\left(x_{n+1} - u\right) \rangle.$$

Since x_n is bounded, there exists a subsequence x_{n_k} of x_n such that $x_{n_k} \rightharpoonup z$ and

$$b = \langle \left(f\left(u\right) - u\right), j\left(x_{n_k+1} - u\right) \rangle.$$

Using (3.9), (3.10), (3.13) and (3.14), we have

$$\begin{aligned} \|x_n - T_{i-1}x_n\| &= \|x_n - x_{n+1} + x_{n+1} - z_n + z_n - T_{i-1}z_n + T_{i-1}z_n - T_{i-1}x_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| + \|z_n - T_{i-1}z_n\| + \|T_{i-1}z_n - T_{i-1}x_n\| \\ \lim_{n\to\infty} \|x_n - T_{i-1}x_n\| &\leq \lim_{n\to\infty} \|x_n - x_{n+1}\| + \lim_{n\to\infty} \|x_{n+1} - z_n\| + \lim_{n\to\infty} \|z_n - T_{i-1}z_n\| \\ &\quad + \lim_{n\to\infty} \|z_n - x_n\|. \end{aligned}$$

Hence, we have

$$\lim_{n\to\infty} \|x_n - T_{i-1}x_n\| = 0 \tag{3.15}$$

Since $\lim\limits_{n\to\infty} \|x_n - T_{i-1}x_n\| = 0$, we conclude that

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0. \tag{3.16}$$

We show that the solution of the variational inequality is unique.
Assume $u, z \in \bigcap\limits_{i=1}^{N} A(T_i)$ are the solution of the variation inequality of (3.5). Then

$$\langle u - f(u), j(z-u)\rangle \geq 0 \qquad\qquad \langle z - f(z), j(u-z)\rangle \geq 0.$$

Adding the two equations, we have

$$\begin{aligned}
0 &\geq \langle u - f(u), j(u-z)\rangle + \langle f(z) - z, j(u-z)\rangle \\
&= \langle u - f(u) + f(z) - z, j(u-z)\rangle \\
&= \langle u - z, -(f(u) - f(z)), j(u-z)\rangle \\
&= \langle u - z, j(u-z)\rangle - \langle f(u) - f(z), j(u-z)\rangle \\
&\geq \|u-z\|^2 - |\langle f(u) - f(z), j(u-z)\rangle| \\
&\geq \|u-z\|^2 - \|f(u) - f(z)\|\|u-z\| \qquad (3.17)\\
&\geq \|u-z\|^2 - K\|u-z\|\|u-z\| \\
&= \|u-z\|^2 - K\|u-z\|^2 \\
&= (1-K)\|z-u\|^2 \\
&= \|z-u\|^2 \\
0 &= \|z-u\|.
\end{aligned}$$

We obtain $z = u$ so the solution is unique.

Since T is a generalized nonexpansive mapping then by Lemma 2.3 and (3.16), we have $z \in \bigcap\limits_{i=1}^{N} A(T_i)$. From (3.5), the following holds:

$$\langle f(u) - u, j(z-u)\rangle \leq 0.$$

Now, we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle f(u) - u, j(x_{n+1} - u)\rangle &= \limsup_{n\to\infty}\langle f(u) - u, j(x_{n+1} - x_n + x_n - u)\rangle \\
&= \limsup_{n\to\infty}\langle f(u) - u, j(x_{n+1} - x_n) + j(x_n - u)\rangle \\
&\leq \limsup_{n\to\infty}\langle f(u) - u, j(x_{n+1} - x_n)\rangle \\
&\quad + \limsup_{n\to\infty}\langle f(u) - u, j(x_n - u)\rangle.
\end{aligned}$$

Since by (3.17) then there exist a subsequence x_{n_k} of x_n such that $x_{n_k} \rightharpoonup z$. We have

$$\begin{aligned}
\limsup_{n\to\infty}\langle f(u) - u, j(x_{n+1} - u)\rangle &\leq \lim_{k\to\infty}\langle f(u) - u, j(x_{n_k+1} - x_{n_k})\rangle \\
&\quad + \lim_{k\to\infty}\langle f(u) - u, j(x_{n_k} - u)\rangle \\
&= \langle f(u) - u, j(z-z)\rangle + \langle f(u) - u, j(z-u)\rangle \\
\limsup_{n\to\infty}\langle f(u) - u, j(x_{n+1} - u)\rangle &\leq \langle f(u) - u, j(z-u)\rangle \leq 0.
\end{aligned}$$

Hence

$$\limsup_{n\to\infty}\langle f(u)-u, j(x_{n+1}-u)\rangle \quad \leq 0. \tag{3.18}$$

By (3.18), with $\lim\limits_{n\to\infty}\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|=0$, we have $\lim\limits_{n\to\infty}\sup t_n\leq 0$. Hence we conclude that $\{x_n\}$ converges strongly to $u\in\bigcap\limits_{i=1}^{N}A(T_i)$.

In the second case, we assume that there exists a subsequence $\{\Phi_{n_t}\}$ of $\{\Phi_n\}$ such that $\Phi_{n_t}<\Phi_{n_t+1}$ for all $t\in N$. Then, we define $\tau:\{n:n\geq n_0\}\to N$ as follows:

$$\tau(n):=\max\{s\in N: s\leq n, \Phi_s<\Phi_{s+1}\}.$$

Obviously, τ is a nondecreasing sequence. Then, by Lemma 2.7, we have $\Phi_{\tau(n)}\leq\Phi_{\tau(n)+1}$, that is $\|x_{\tau(n)}-u\|\leq\|x_{\tau(n)+1}-u\|$ for all $n\geq n_0$. Using similar argument as in the first case, we can obtain everything proved in the first case by taking $\tau(n)$ instead of n. So we have

$$\limsup_{n\to\infty}\|x_{\tau(n)}-u\|^2\leq 0.$$

Therefore, we get

$$\|x_{\tau(n)}-u\|\to 0 \text{ and } \|x_{\tau(n)+1}-u\|\to 0 \quad \text{as} \quad n\to\infty. \tag{3.19}$$

So, it follows from (3.19) and Lemma 2.7 that

$$\|x_n-u\|\leq\|x_{\tau(n)+1}-u\|\to 0 \quad \text{as} \quad n\to\infty.$$

Therefore, $\{x_n\}$ converges strongly to $u\in\bigcap\limits_{i=1}^{N}A(T_i)$.

□

4 Numerical Example

In this section, we present a numerical example to illustrate the behavior of the sequences generated by the iterative scheme (3.1). The numerical implementation is done with the aid of MATLAB R2017a programming on a PC.

Example 4.1. *Let $X=R$ with the usual norm $(X,|.|)$. Suppose $T_i=X\to X$, for every $i=1,2,\ldots,N$ be defined by*

$$T_i(x)=\begin{cases}\frac{x}{3+i} & if \quad x<1\\ \frac{x}{(3+i)^2} & if \quad x\geq 1.\end{cases}$$

We take $\alpha_n=\frac{1}{50n-1}$, $\beta_n=\frac{n}{50n-1}$, $\Omega_n=1-\alpha_n-\beta_n$, $\gamma_n=\frac{1}{2^n}$, $\theta_n=\frac{1}{n^2+1}$ and $f(x)=\frac{x}{2}$, where f is a contraction mapping with $K\in[0,1)$ then all the conditions are satisfied. Then from (3.1) we get

$$\begin{cases}x_0,x_1\in X\\ z_n=x_n+\frac{1}{n^2+1}(x_n-x_{n-1}) \qquad n\geq 1\\ y_n=\sum\limits_{i=1}^{N+1}\frac{1}{2^n}(T_{i-1}z_n) \qquad \forall \quad i=1,2,3,\ldots,N+1\\ x_{n+1}=\frac{1}{50n-1}\left(\frac{x_n}{2}\right)+\left(\frac{n}{50n-1}\right)x_n+\left(\frac{49n-2}{50n-1}\right)y_n \qquad \forall \quad n\geq 1.\end{cases} \tag{4.1}$$

We test the iterative methods for the following initial points:

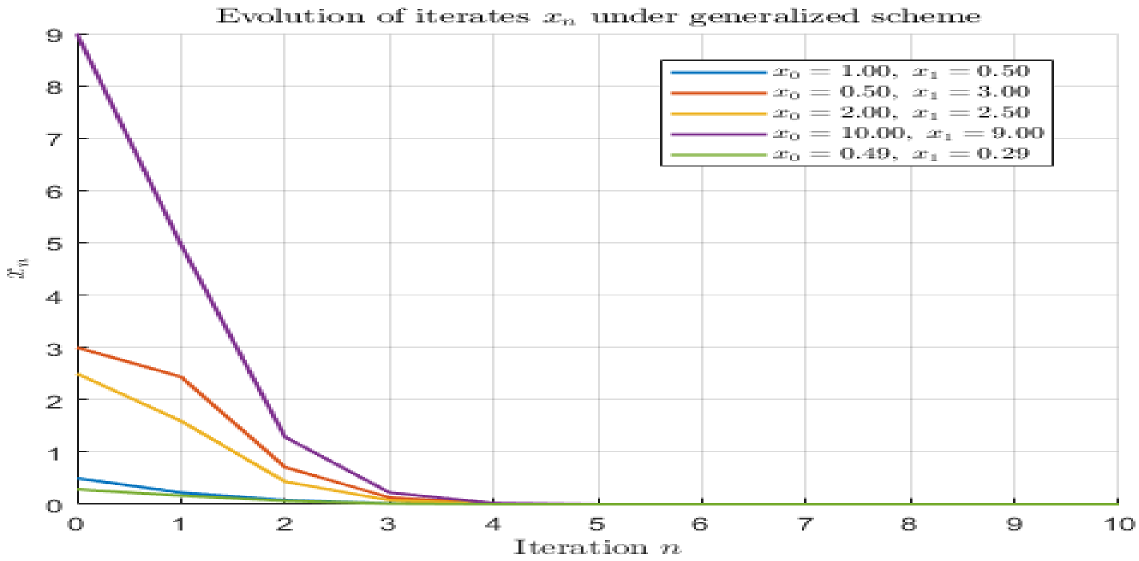
1. $x_0=1.00$ and $x_1=0.5$

2. $x_0 = 0.50$ and $x_1 = 3.00$
3. $x_0 = 2.00$ and $x_1 = 2.5$
4. $x_0 = 10.00$ and $x_1 = 9.00$
5. $x_0 = 0.488$ and $x_1 = 0.288$

The values of $\{x_n\}$ for different initial guesses

Ition no.x_n	case 1	case 2	case 3	case 4	case 5
1	0.50000	3	2.50000	9.00000	0.288
2	0.22627	2.437239	1.59414	4.96631	0.16746
3	0.07888	0.710028	0.43441	1.28574	0.06538
4	0.01558	0.13168	0.07836	0.22662	0.01334
5	0.00162	0.01344	0.00793	0.02274	0.00140
6	9.41738×10^{-5}	7.7463×10^{-4}	4.55134×10^{-4}	0.00130	8.17196×10^{-5}
7	3.46596×10^{-6}	2.84413×10^{-5}	1.66918×10^{-5}	4.76948×10^{-5}	3.01099×10^{-6}
8	9.66689×10^{-8}	7.92763×10^{-7}	4.65120×10^{-7}	1.32868×10^{-6}	8.40043×10^{-8}
9	2.36056×10^{-9}	1.93567×10^{-8}	1.13562×10^{-8}	3.24391×10^{-8}	2.05140×10^{-9}
10	5.40122×10^{-11}	4.42898×10^{-10}	2.59839×10^{-10}	7.42232×10^{-10}	4.69384×10^{-11}

It is evident from Table above that $\{x_n\} \to 0 \in \bigcap_{i=1}^{N} A(T_i)$.



5 Conclusion

We have studied the convergence of attractive points of finitely many families of generalized nonexpensive mappings using viscosity approximation method together with inertia parameters in the setting of uniformly convex Banach space. Our theorem extends the results of [15] from two to finitely many families of the said mappings. We have also established strong convergence theorem without the so called condition A and compactness assumption on the domain.

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