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Fixed Point Results for Some Enriched Contractions in Convex B-Metric Spaces

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Abstract

In this study, we show the existence of fixed points of enriched contractions and enriched ϕ-contractions of Hardy-Rogers type using Krasnoselskii's iterative process. Furthermore, we show that our results imply related fixed-point results for enriched contractions and enriched ϕ-contractions of Banach, Kannan, Chatterjea and Reich types. Our results are novel for convex b-metric spaces and generalize several established results in convex metric spaces.

Keywords: Fixed point, Krasnoselskii's iterative method, Hardy-Rogers contraction, B-metric spaces.
MSC2010: 47H09, 47H10.

1 Introduction and Preliminaries

In [1] it was demonstrated that Picard's iterative scheme may fail to converge to fixed points of non-expansive mappings. To address this weakness, iterative schemes, such as Krasnoselkii's iterative scheme developed in [2], were formulated. However, the convergence of such iterative schemes depends on the convexity of the ambient space.

In order to adapt convexity-dependent iterative methods for applications to problems in metric spaces, [3] defined convex structures in metric spaces. Metric spaces in which such convex structures are defined are called convex metric spaces. ([3]): Let (X,d) be a metric space and $W : X \times X \times [0,1] \to \mathbb{R}$ be a mapping such that for all $u,x,y \in X$ and $\lambda \in [0,1]$,

$$d(u, W(x,y;\lambda)) \leq \lambda d(u,x) + (1-\lambda)d(u,y).$$

Then W is called a convex structure on X and (X,d,W) is called a convex metric space.

The existence of fixed points of certain contraction mappings in convex metric spaces was shown in [3] and [4]

[5] defined b-metric spaces which are extensions of metric spaces, and [6] defined convex b-metric spaces, thereby extending the notion of convexity to b-metric spaces. The respective definitions

are as follows: ([5]): Let X be a non-empty set, s a positive real number, and $d_b : X \to [0, \infty)$ a mapping satisfying the following properties $\forall x, y, z \in X$.
(i) $d_b(x, y) = 0 \iff x = y$;
(ii) $d_b(x, y) = d_b(y, x)$;
(iii) $d_b(x, y) \leq s[d_b(x, z) + d_b(z, y)]$.
Then (X,d_b) is called a b-metric space with coefficient s.

([6]): Let (X, d_b) be a b-metric space and $W : X \times X \times [0, 1] \to X$ be a mapping such that

$$d_b(u, W(x, y; \lambda)) \leq \lambda d_b(u, x) + (1 - \lambda)d_b(u, y), \tag{1}$$

$\forall(u, x, y, \lambda) \in X \times X \times X \times [0, 1]$, then (X, d_b, W) is called a convex b-metric space, and W is called a convex structure on (X, d_b). The existence of fixed points of certain contraction mappings in convex b-metric spaces was shown in [6] and [7]

To generalize the contraction mapping in [8], [9] and [10] gave definitions of new contraction-type mappings and proved related fixed-point results for them.

Theorem 1.1. *([9]): Let (M,d) be metric space and T a self-mapping of M satisfying the following for $x, y \in M$*

$$d(Tx, Ty) \leq ad(x, Tx) + bd(y, Ty) + cd(x, Ty) + ed(y, Tx) + fd(x, y), \tag{2}$$

where a,b,c,e and f are nonnegative. Let $\alpha = a + b + c + e + f$, then if M is complete and $\alpha < 1$, T has a unique fixed point.

([10]) Let (X,d,W) be a convex metric space. A mapping $T : X \to X$ is said to be an enriched contraction if there exists $c \in [0, 1)$ and $\lambda \in [0, 1]$ such that

$$d(W(x, Tx; \lambda), W(y, Ty; \lambda)) \leq cd(x, y), \tag{3}$$

for all $x, y \in X$ A mapping satisfying (3) is called a (c, λ)-enriched contraction.

From the following lemma, it is clear that a $(0, c)$-enriched contraction is a Banach contraction($d(Tx, Ty) \leq cd(x, y), c \in [0, 1)$)

Lemma 1.2. *([10]) Let (X,d, W) be a convex metric space. For each $x, y \in X$ and $\lambda \in [0, 1]$, we have the following.*
(i) $W(x, x; \lambda) = x$;
(ii) $W(x, y; 0) = y$;
(iii) $W(x, y; 1) = x$.

From what has been discussed thus far, it can be seen that an enriched contraction is defined in a metric space only if a convex structure is defined therein. This is because for any metric space, an enriched contraction in the space is defined in terms of an upper bound on the metric distance between the images of any two points in the metric space under the mapping defined by the convex structure. This fact makes iterative methods for which iterative steps are defined in terms of convex structures ideal for proving the existence of fixed points of enriched contractions. The krasnoselskii's iterative scheme is such a method.

The following is a fixed-point result in [10] for (λ, c)-enriched contractions in metric spaces.

Theorem 1.3. *([10]) Let (X,d, W) be a complete metric space and let $T : X \to X$ be a (λ, c)-enrinched contraction. Then,*
(i) Fix(T) = p, for some $p \in X$;
(ii) The sequence $\{x_n\}$ obtained from the iteration process

$$x_{n+1} = W(x_n, Tx_n; \lambda), \quad n \geq 1, \tag{4}$$

converges to p, for any $x_0 \in X$.

The sequence defined by (4) is the Krasnoselskii's sequence.

Also, [10] defined enriched ϕ-contractions using a comparison function ϕ,and establish a related fixed-point result. ([10]): A mapping $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a comparison function if it satisfies the following conditions:

(i_ϕ) ϕ is nondecreasing;
(ii_ϕ) $\{\phi^n(t)\}$ convergences to 0 $\forall t \geq 0$;
(iii_ϕ) $\phi(t) < t$, for $t > 0$.

Theorem 1.4. *([10]): Let (X,d, W) be a complete metric space and let $T : X \to X$ be an enriched ϕ-contraction, i.e., a mapping for which there exists a comparison function ϕ such that for some $\lambda \in [0, 1)$,*

$$d(W(x, Tx; \lambda), W(y, Ty; \lambda)) \leq \phi(d(x, w)),$$

$\forall x, y \in X$. Then:
(i) fix(T) = $\{p\}$, for some $p \in X$;
(ii) The sequence $\{x_n\}$ obtained from the Iterative process

$$x_{n+1} = W(x_n, Tx_n; \lambda), \quad n \geq 1,$$

converges to p, for any $x_0 \in X$.

In this work, we generalize the results in [10] and [6] by showing the existence of fixed points of enriched contractions and enriched ϕ-contractions of Hardy-Rogers type. Furthermore, we show that special cases of our results generalize established results for several other contractions.

2 Main Results

Theorem 2.1. *Let (X, d_b, W) be a complete convex b-metric space with constant $s \geq 1$ and $\lambda \in [0, \frac{1}{2s}]$. Then the sequence $\{x_n\}$ defined by*

$$x_{n+1} = W(x_n, Tx_n; \lambda), \quad x_0 \in X,$$

converges to a unique fixed point of a mapping $T : X \to X$ satisfying the following condition for any $x, y \in X$

$$d_b(W(x, Tx; \lambda), d_b(W(y, Ty; \lambda)) \leq ad_b(x, y) + bd_b(x, Tx) + cd_b(y, Ty) + ed_b(x, Ty) + fd_b(y, Tx)$$

, where a,b,c,e,and f are non-negative numbers such that

$$a + b + c + 2se + 2sf + \lambda < \frac{1}{2s}.$$

Proof: First, we show that the Krasnoselskii's sequence associated with the mapping is Cauchy.

$$\begin{aligned} d_b(x_n, x_{n+1}) &= d_b(x_n, W(x_n, Tx_n; \lambda) \\ &\leq \lambda d_b(x_n, x_n) + (1-\lambda)d_b(x_n, Tx_n) \\ &\leq (1-\lambda)d_b(x_n, Tx_n) \leq d_b(x_n, Tx_n). \qquad (5) \end{aligned}$$

$$\begin{aligned} d_b(x_n, Tx_n) &= d(W(x_{n-1}, Tx_{n-1}; \lambda), Tx_n) \\ &\leq s[d_b(W(x_{n-1}, Tx_{n-1}; \lambda), W(x_n, Tx_n; \lambda)) + d_b(W(x_n, Tx_n; \lambda), Tx_n)] \\ &\leq s[ad_b(x_{n-1}, x_n) + bd_b(x_{n-1}, Tx_{n-1}) + cd_b(x_n, Tx_n) \\ &+ ed_b(x_n, Tx_{n-1}) + fd_b(x_{n-1}, Tx_n) \\ &+ \lambda d_b(x_n, Tx_n) + (1-\lambda)d_b(Tx_n, Tx_n] \\ &= s[ad_b(x_{n-1}, x_n) + bd_b(x_{n-1}, Tx_{n-1}) + cd_b(x_n, Tx_n) \\ &+ ed_b(x_n, Tx_{n-1}) + fd_b(x_{n-1}, Tx_n) + \lambda d_b(x_n, Tx_n)] \qquad (5^*) \\ \implies (1 - s(c+\lambda))d_b(x_n, Tx_n) &\leq s[ad_b(x_{n-1}, x_n) + bd_b(x_{n-1}, Tx_{n-1}) \\ &+ ed_b(x_n, Tx_{n-1}) + fd_b(x_{n-1}, Tx_n)]. \qquad (6) \end{aligned}$$

$$\begin{aligned} d_b(x_{n-1}, x_n) &= d_b(x_{n-1}, W(x_{n-1}, Tx_{n-1}; \lambda) \\ &\leq \lambda d_b(x_{n-1}, x_{n-1}) + (1-\lambda) d_b(x_{n-1}, T_{n-1}) \\ &= (1-\lambda) d_b(x_{n-1}, Tx_{n-1}) \leq d_b(x_{n-1}, Tx_{n-1}). \end{aligned} \tag{7}$$

$$\begin{aligned} d_b(x_n, Tx_{n-1}) &\leq s(d_b(x_n, x_{n-1}) + d_b(x_{n-1}, Tx_{n-1}) \\ &\leq s(d_b(x_{n-1}, Tx_{n-1}) + d(x_{n-1}, Tx_{n-1}) \\ &= 2sd_b(x_{n-1}, Tx_{n-1}). \end{aligned} \tag{8}$$

$$\begin{aligned} d_b(x_{n-1}, Tx_n) &\leq s[d_b(x_{n-1}, x_n) + d_b(x_n, Tx_n)] \\ &\leq s[d_b(x_{n-1}, Tx_{n-1}) + d_b(x_n, Tx_n)]. \end{aligned} \tag{9}$$

substituting (7),(8) and (9) into (6), we obtain

$$\begin{aligned} (1 - s(c + f + \alpha_n)) d_b(x_n, Tx_n) &\leq (sa + sb + 2s^2 e + s^2 f) d_b(x_{n-1}, Tx_{n-1}) \\ \implies d_b(x_n, Tx_n) &\leq \frac{sa + sb + 2s^2 e + s^2 f}{1 - s(c + f + \lambda)} d_b(x_{n-1}, Tx_{n-1}). \end{aligned} \tag{10}$$

Let $k = \frac{sa+sb+2s^2e+s^2f}{1-s(c+f+\lambda)}$, then since

$$\frac{sa + sb + 2es^2 + s^2 f}{1 - s(c + f + \lambda)} < \frac{1}{s} \leq 1,$$

it follows that $k < 1$

From (10)

$$\begin{aligned} d_b(x_n, Tx_n) \leq k_n d_b(x_{n-1}, Tx_{n-1}) &\leq k_n . k_{n-1} d_b(x_{n-2}, Tx_{n-2}) \\ &\leq \ldots \leq \prod_{i=1}^{n} k_i d(x_0, Tx_0). \end{aligned} \tag{11}$$

from (11) and (5),

$$d_b(x_n, x_{n+1}) \leq \prod_{i=1}^{n} k_i d_b(x_0, Tx_0) \tag{12}$$

$$\implies d_b(x_{n+p-1}, x_{n+p}) \leq \prod_{i=1}^{n+p-1} k_i d_b(x_0, Tx_0). \tag{13}$$

where $p \in \mathbb{Z}_+$

from (12), (13) and repeated applications of the triangle-inequality property, we get

$$\begin{aligned} d_b(x_n, x_{n+p}) &\leq sd_b(x_n, x_{n+1}) + sd_b(x_{n+1}, x_{n+p}) \\ &\leq sd_b(x_n, x_{n+1}) + s^2 d_b(x_{n+1}, x_{n+2}) + s^2 d_b(x_{n+2}, x_{n+p}) \\ &\leq sd_b(x_n, x_{n+1}) + s^2 d_b(x_{n+1}, x_{n+2}) \\ &+ s^3 d_b(x_{n+2}, x_{n+3}) + \ldots + s^p d_b(x_{n+p-1}, x_{n+p}) \\ &\leq d_b(x_0, Tx_0) \left[s \prod_{i=1}^{n} k_i + s^2 \prod_{i=1}^{n+1} k_i + \ldots + s^p \prod_{i=1}^{n+p-1} k_i \right]. \end{aligned} \tag{14}$$

Let $U_{n_k} = s\prod_{i=1}^{n+k} k_i$, where k is a non-negative integer such that $0 \leq k \leq p-1$

$$\lim_{n\to\infty} \frac{U_{n_{k+1}}}{U_{n_k}} = \lim_{n\to\infty} k_n = k < 1.$$

By ratio test, $\sum_{n=1}^{\infty} U_{n_k}$ is convergent, and it follows that the sequence $\{U_{n_k}\}$ converges to 0. Accordingly, as $n \to \infty$, the RHS of (14) tends to 0 . Therefore

$$\lim_{n\to\infty} d_b(x_n, x_{n+p}) = 0.$$

This shows that $\{x_n\}$ is a Cauchy sequence in the space and so converges to a point, say x^*, in the space, since the space is complete. Therefore, $\lim_{n\to\infty} d_b(x_n, x*) = 0$. Using a similar argument and taking $v_j = \prod_{i=0}^{j-1} k_i$, we get $\lim_{n\to\infty} d_b(x_n, Tx_n) = 0$ from (11). Thus,

$$\lim_{n\to\infty} d_b(x_n, x^*) = \lim_{n\to\infty} d_b(x_n, Tx_n) = 0. \tag{15}$$

Next, we show that x^* which is the limit of the sequence generated by the iterative process is a fixed point of T.

$$\begin{aligned}
d_b(x^*, Tx^*) &\leq s[d_b(x^*, Tx_n) + d_b(Tx_n, Tx^*)] \\
&\leq s[d_b(x^*, Tx_n) + d_b(w(x_n, Tx_n; 0), w(x^*, Tx^*; 0)] \\
&\leq s[s(d_b(x^*, x_n) + d_b(x_n, Tx_n)) + ad_b(x^*, x_n) + bd_b(x_n, Tx_n) + cd_b(x^*, Tx^*) \\
&+ ed_b(x_n, Tx^*) + fd_b(x^*, Tx_n)] \\
&\leq s[s(d_b(x^*, x_n) + d_b(x_n, Tx_n)) + ad_b(x^*, x_n) + bd_b(x_n, Tx_n) + cd_b(x^*, Tx^*) \\
&+ se(d_b(x_n, x^*) + d_b(x^*, Tx^*)) + sf(d_b(x*, x_n) + d_b(x_n, Tx_n))].
\end{aligned} \tag{16}$$

Taking limit as $n \to \infty$, (16) becomes

$$\begin{aligned}
d(x^*, Tx^*) &\leq (sc + s^2e)d_b(x^*, Tx^*) \\
\implies (1 - (sc{+}s^2e))&d_b(x^*, Tx^*) \leq 0.
\end{aligned} \tag{17}$$

From hypothesis, $c + se < \frac{1}{s} \implies sc + s^2e < 1 \implies 1 - (sc + s^2e) > 0$. Hence (17) implies that $d_b(x^*, Tx^*) \leq 0$. But $d_b(x^*, Tx^*) \geq 0$, therefore $d_b(x^*, Tx^*) = 0 \implies Tx^* = x^*$. Hence, x^* is a fixed point of T in the space. Next, we show the uniqueness of x^* as a fixed point of T. Let y^* be a different fixed point of T in the space, then $d_b(x^*, y^*) > 0$.

$$\begin{aligned}
d_b(x^*, y^*) = d_b(Tx*, Ty^*) &\leq d_b(w(x^*, Tx^*; 0), w(y^*, Ty^*; 0)) \\
&\leq ad_b(x*, y*) + bd_b(x*, Tx*) + cd_b(y*, Ty^*) + ed_{(}x^*, Ty^*) + fd_b(y^*, Tx^*) \\
\implies (1 - (a{+}e + f))d_b(x^*, y^*) \leq 0.&
\end{aligned} \tag{18}$$

From hypothesis, it follows that $a{+}e{+}f < 1$. Hence (18) implies that $d_b(x*, y^*) \leq 0$, a contradiction.

Therefore x^* is a unique fixed point of T in the space. □

Remark 2.2. *In Theorem 3.1,*

(i) taking $a \neq 0$ and $b = c = e = f = 0$, we get a result for enriched Banach-type contractions in convex b-metric spaces. This generalizes the related result in the work of [10];

(ii) taking $b = c \neq 0$, and $a = f = e = 0$, we get a result for enriched Kannan-type contractions in convex b-metric spaces. This generalizes the related result in the work of [12]

(iii) taking $e = f \neq 0$ and $a = b = c = 0$, we get a result for enriched Chatterjea -type contractions in convex b-metric spaces. This generalizes the related result in the work of [13].

(iv) taking $e = f = 0$, and $a \neq 0, b \neq 0$,and $c \neq 0$ we get a result for enriched Reich -type contractions. This generalizes the related result in the work of ([11]).

Theorem 2.3. *Let* (X, d_b, W) *be a complete convex b-metric space with constant* $s \geq 1$ *and* $\lambda \in [0, \frac{1}{2s}]$*. Then the sequence* $\{x_n\}$ *defined by*

$$x_{n+1} = W(x_n, Tx_n; \lambda), \quad x_0 \in X, x_0 \neq Tx_0,$$

converges to a unique fixed point of a mapping $T : X \to X$ *satisfying the following condition for any* $x, y \in X$

$$d_b(W(x, Tx; \lambda), d_b(W(y, Ty; \lambda)) \leq \phi[ad_b(x, y) + bd_b(x, Tx) + cd_b(y, Ty) + ed_b(x, Ty) + fd_b(y, Tx),]$$

where a,b,c,e,*and* f *are non-negative numbers such that*

$$a + b + c + 2se + 2sf + \lambda < \frac{1}{2s}$$

and $(max\{c, f\}, max\{a, b, e, f\}) \in \mathbb{Z}_+ \times \mathbb{Z}_+$.

Proof: First, we show that the Krasnoselskii sequence associated with the contraction mapping is Cauchy. From (5),

$$d_b(x_n, x_{n+1}) \leq d_b(x_n, Tx_n). \tag{19}$$

Using the same technique through which (5*) was derived, we get

$$\begin{aligned} d_b(x_n, Tx_n) \leq s[&\phi[ad_b(x_{n-1}, x_n) + bd_b(x_{n-1}, Tx_{n-1}) + cd_b(x_n, Tx_n) \\ &+ ed_b(x_n, Tx_{n-1}) + fd_b(x_{n-1}, Tx_n)] + \lambda d_b(x_n, Tx_n). \end{aligned} \tag{20}$$

substituting (7),(8) and (9) into (20), we get:

$$\begin{aligned} d_b(x_n, Tx_n) \leq \phi[&(s(c + f)d_b(x_n, Tx_n) + (sa + sb + 2s^2e + s^2f)d_b(x_{n-1}, Tx_{n-1}) \\ &+ s\lambda d_b(x_n, Tx_n)]. \end{aligned} \tag{21}$$

Clearly,

$$s(c + f)d_b(x_n, Tx_n) + (sa + sb + 2s^2e + s^2f)d_b(x_{n-1}, Tx_{n-1}) \geq 0. \tag{22}$$

Suppose

$$s(c + f)d_b(x_n, Tx_n) + (sa + sb + 2s^2e + s^2f)d_b(x_{n-1}, Tx_{n-1}) = 0. \tag{23}$$

Now,

(i) $c = f = 0$ contradicts the hypothesis that $max\{c, f\} > 0$;

(ii) $d_b(x_n, Tx_n) = 0 \quad \forall n \in \mathbb{Z}_+ \implies d_b(x_0, Tx_0) = 0$, which contradicts the hypothesis that $x_0 \neq Tx_0$;

(iii) $a = b = e = f = 0$ contradicts the hypothesis that $max\{a, b, e, f\} > 0$;

(iv) $d_b(x_{n-1}, Tx_{n-1}) = 0 \quad \forall n \in \mathbb{Z}_+ \implies d_b(x_0, Tx_0) = 0$, which is a contraction to the hypothesis that $Tx_0 \neq x_0$

Hence (23) does not hold and from (22) we get that

$$s(c + f)d_b(x_n, Tx_n) + (sa + sb + 2s^2e + s^2f)d_b(x_{n-1}, Tx_{n-1}) > 0. \tag{24}$$

Let t be the LHS of (24), then we have $t > 0$, and from property (iii_ϕ) of ϕ, this implies that

$$\phi(t) < t. \tag{25}$$

Substituting (25) into (21) gives

$$\begin{aligned} d_b(x_n, Tx_n) &\leq s(c + f)d(x_n, Tx_n) + (sa + sb + 2s^2e + s^2f)d_b(x_{n-1}, Tx_{n-1}) \\ &\quad s\lambda d_b(x_n, Tx_n) \\ \implies d_b(x_n, Tx_n) &\leq \frac{sa + sb + 2s^2e + s^2f)}{1 - s(c + f + \lambda)} d_b(x_{n-1}, Tx_{n-1}). \end{aligned} \tag{26}$$

(26) and (10) in Theorem 3.1 are similar, and so showing that $x_n \to x^*$, where $x^* \in X$, and $\lim\limits_{n\to\infty} d_b(x_n, Tx_n) = 0$ follows from the same argument used to prove (15) in Theorem 3.1

Accordingly, it now suffices to show that x^* is a fixed point of T and a unique one at that.

$$\begin{aligned} d_b(x^*, Tx^*) &\leq s[d_b(x^*, Tx_n) + d_b(Tx_n, Tx^*)] \\ &\leq s[d_b(x^*, Tx_n) + d_b(w(x_n, Tx_n; 0), w(x^*, Tx^*; 0)] \\ &\leq s[s(d_b(x^*, x_n) + d_b(x_n, Tx_n)) + \phi[ad_b(x^*, x_n) + bd_b(x_n, Tx_n) + cd_b(x^*, Tx^*) \\ &+ ed_b(x_n, Tx^*) + fd_b(x^*, Tx_n)]]. \end{aligned} \tag{27}$$

from (25), (27) becomes

$$\begin{aligned} d_b(x*, Tx^*) \leq s[&s(d_b(x^*, x_n) + d_b(x_n, Tx_n)) + ad_b(x^*, x_n) + bd_b(x_n, Tx_n) + cd_b(x^*, Tx^*) \\ &+ se(d_b(x_n, x^*) + d_b(x^*, Tx^*)) + sf(d_b(x*, x_n) + d_b(x_n, Tx_n))]. \end{aligned} \tag{28}$$

As $n \to \infty$, (28) becomes

$$\begin{aligned} d(x^*, Tx^*) &\leq (sc + s^2e)d_b(x^*, Tx^*) \\ \implies (1 - (sc{+}s^2e))d_b(x^*, Tx^*) &\leq 0. \end{aligned} \tag{29}$$

From hypothesis, $c + se < \frac{1}{s} \implies sc + s^2e < 1 \implies 1 - (sc + s^2e) > 0$. Hence (29) implies that $d_b(x^*, Tx^*) \leq 0$. But $d_b(x^*, Tx^*) \geq 0$, therefore $d_b(x^*, Tx^*) = 0 \implies Tx^* = x^*$. Hence, x^* is a fixed point of T in the space.

Next, we show the uniqueness of x^* as a fixed point of T. Let y^* be a different fixed point of T in the space, then $d_b(x^*, y^*) > 0$.

$$\begin{aligned} d_b(x^*, y^*) = d_b(Tx*, Ty^*) &\leq d_b(w(x^*, Tx^*; 0), w(y^*, Ty^*; 0)) \\ &\leq \phi[ad_b(x*, y*) + bd_b(x*, Tx*) + cd_b(y*, Ty^*) + ed_{(}x^*, Ty^*) + fd_b(y^*, Tx^*)] \\ &\leq ad_b(x*, y*) + bd_b(x*, Tx*) + cd_b(y*, Ty^*) + ed_{(}x^*, Ty^*) + fd_b(y^*, Tx^*) \quad (from (25)) \\ &\implies (1 - (a + e + f))d_b(x^*, y^*) \leq 0. \end{aligned} \tag{30}$$

From hypothesis, it follows that $a{+}e{+}f < 1$. Hence (30) implies that $d_b(x*, y^*) \leq 0$, a contradiction.

Therefore x^* is a unique fixed point of T in the space. □

Remark 2.4. *In Theorem 3.2,*

(i) by taking $a \neq 0$ and $b = c = e = f = 0$, we get a result for enriched ϕ-contraction of Banach type in convex b-metric spaces. This generalizes the related result in the work of [10].

(ii) by taking $b = c \neq 0$, and $a = f = e = 0$, we get a result for enriched ϕ-contractions of Kannan type in b-metric spaces. This generalizes the related result in the work of [12].

(iii) by taking $e = f \neq 0$ and $a = b = c = 0$, we get a result for enriched ϕ-contractions of Chatterjea-type in convex b-metric spaces.This generalizes the related result in the work of [13].

(iv) by taking $e = f = 0$, and $a \neq 0, b \neq 0$,and $c \neq 0$ we get a result for enriched $\phi-$ contactions of Reich type in convex b-metric spaces. This generalizes the related result in the work of [11]

Conclusion:

We have established fixed-point results for enriched Hardy-Rogers contractions and an enriched $\phi-$Hardy-Rogers contractions in convex b-metric spaces. Also, we have shown that our results imply similar results for other enriched contractions in the same framework.

Conflict of Interest:

None

References

[1] V. Berinde, iterative approximation of fixed points, 2nd ed. Springer, 2007, pp. 15–16.

[2] M. Krasnoselskii, "Two remarks on the method of successive approximations," uspehi Mat. Nauk., vol. 10, no. 1, pp. 123–127, 1955.

[3] W. Takahashi, "Convexity in metric space and non-expansive mappings," Kodai Math. Sem. Rep., vol. 22, no. 2, pp. 142–149, 1970. [Online]. Available: https://doi.org/10.2996/kmj/1138846111.

[4] O. J. Omidire, "Common fixed point theorems of some certain generalized contractive conditions in convex metric space settings.," International Journal of Mathematical Sciences and Optimization: Theory and Applications, vol. 10, no. 3, pp. 1–9, 2024. [Online]. Available: https: //doi.org/10.6084/zenodo.13152753.

[5] S. Czwewik, "Contraction mappings in b-metric spaces," Acta Mathematica et Informatica Universitatis Ostraviensis, vol. 1, no. 1, pp. 5–11, 1993.

[6] L. Chen, C. Li, R. Kaczmarek, and Y. Zhao, "Several fixed point theorems in convex b-metric spaces and applications," Mathematics, vol. 8, p. 242, 2020. [Online]. Available: https://doi.org/10.3390/math8020242.

[7] G. Akinbo and O. O. Fabelurin, "Convergence of implicit noor iteration in convex b-metric space.," International Journal of Mathematical Sciences and Optimization: Theory and Applications, vol. 11, no. 1, pp. 89–95, 2025.

[8] S. Banach, "Sur les op´erations dans les ensembles abstraits et leurs applications aux ´equations int´egrales," Fundamental Mathematics, vol. 3, pp. 133–181, 1922.

[9] G. Hardy and T. Rogers, "A generalization of fixed point theorem of Reich," Canada Math. Bull., vol. 16, no. 2, pp. 201–206, 1973. [Online]. Available: https://doi.org/10.4153/CMB-1973-036-0.

[10] V. Berinde and M. Pacurar, "Existence and approximation of fixed points of enriched contractions and enriched contractions," Symmetry, vol. 13, no. 3, 2021. [Online]. Available: https://doi.org/10.3390/sym13030498.

[11] R. Kannan, "Some results on fixed point," Bull. Calcutta math. soc., vol. 60, 1968.

[12] S. Chatterjea, "Fixed point theorems.," C.R. Acad. Bulgare Sci., vol. 30, pp. 737–740, 1972.

[13] S. Reich and I. Shafrir, "Nonexpansive iterations in hyperbolic spaces," Nonlinear Anal., vol. 15, no. 6, pp. 537–558, 1990. [Online]. Available: https://doi.org/10.1016/0362-546X(90)90058-O.