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On the Solution of Some Vibration Problems Using Variational Iteration Method.

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Abstract

In this paper, we implement a new analytical technique, variational iteration method for solving some vibration problems. The variational iteration algorithm leads to analytical solutions in the present study. The results show that the present method can be easily extended to other nonlinear oscillations and it can be predicted that variational iteration method can be found widely applicable in engineering and physics.

Keywords: Variational Iteration Method (VIM), Vibrations, Analytical solutions and Nonlinear oscillations.
MSC2010: 74H10

1 Introduction

Vibration of dynamical systems can be divided into two main classes like discrete systems depend on time only, whereas in distributed systems such as beams, plates etc. variables depend on time and space. Therefore, equations of motion of discrete systems are described by ordinary differential equations, while equations of motion of distributed systems are described by partial differential equations (Meirovitch [1]).

On the other hand, in the last decades, scientist have proposed and applied some analytical methods to nonlinear equations. For example; the vibrational behavior of quintic nonlinear in extensional beam on two-parameter elastic substrate based on the three mode assumptions is investigated by Sedighi [2]. He employed parameter expansion method to obtain the approximate expressions of nonlinear frequency-amplitude relationship for the first, second and third modes of vibrations. Hamiltonian approach is applied to the analysis of the nonlinear free vibration of a tapered beam by Pakar and Bayat [3]. Ghaffarzadeh and Nikkar [4] applied a new analytical method called the variational iteration method-II (VIM-II) for the differential equation of the large deformation of a cantilever beam under point load at the free tip. Askari et al. [5] applied He's energy balance method and He's variational approach to frequency analysis of nonlinear oscillators with rational restoring force. Sedighi and Daneshmand [6] studied nonlinear transversely vibrating beams by the homotopy perturbation method with an auxiliary term. Jodeiri and Tabrizi [7] employed variational approach to obtain approximate analytical solutions for nonlinear vibrations of a thin laminated composite plate. They also applied Hamiltonian approach to analyze the nonlinear problem of an elastically restrained tapered cantilever beam [8]. Bagheri et al. [9] studied the nonlinear responses of clamped-clamped buckled beam. They used two efficient mathematical techniques called He's variational approach and Laplace iteration method in order to obtain the responses of the beam vibrations. Salehi et al. [10] applied two efficient methods to consider large deformation of cantilever beams under point load. A novel approach is applied to the analytic treatment of nonlinear fifth-order Equations by Saravi and Nikkar [11]. Askari et al. [12] applied higher order Hamilton approach to nonlinear vibrating systems, and many other problems solved by these methods [13] - [16].

The main goal of this paper is to present an alternative approach, namely variational iteration method, for constructing highly accurate analytical approximations to the nonlinear oscillation problems.

In the course of developing the variation method, we introduced some new nomenclatures, such as restricted variation, correction functional. Heuristic interpretation of those concepts leads new comers to the field to start working immediately without the long search and preparation of advanced calculus and calculus of variations, at the same time those already familiar with variation iteration method will find the most recent new results.

Lagrange multiplier

We know Lagrange multiplier well in optimization and calculus of variations. Inokuti *et al.* [16] suggested a method of general Lagrange multiplier . In order to understand the concept of the general Lagrange multiplier, we consider an algebraic equation

$$f(x) = 0, \qquad x \in \mathbb{R} \tag{1.1}$$

If x_n is an approximate root of the above equation, it follows:

$$f(x_n) \neq 0. \tag{1.2}$$

To improve its accuracy, we write the following correction equation:

$$x_{n+1} = x_n + \lambda f(x_n), \tag{1.3}$$

where λ is a general Lagrange multiplier, which can be identified optimally by setting

$$\frac{dx_{n+1}}{dx_n} = 0, \tag{1.4}$$

which leads to the well-known Newton iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1.5}$$

There are alternative approaches to construction of correction equation. We write another correction for x_n as follows:

$$x_{n+1} = x_n + \lambda g(x_n) f(x_n), \tag{1.6}$$

where g(x) is an auxiliary function. After identification of the multiplier, we have a general iteration formulation:

$$x_{n+1} = x_n - \frac{g(x_n) f(x_n)}{g(x_n) f'(x_n) + g'(x_n) f(x_n)} \tag{1.7}$$

The value of the auxiliary function should not be zero or small value during the all iteration steps,$|g(x_n)| > 1$. If we choose $g(x_n) = e^{-\alpha x_n}$, the iteration formulation, equation (1.7), reduces to

$$x_{n+1} = x_n - \frac{f(x_n}{f'(x_n) - \alpha f(x_n)} \tag{1.8}$$

This iteration formulation is very effective when $f'(x_n)$ is small. As an example, we consider the equation

$$\sin x = 0 \tag{1.9}$$

If we begin with $x_0 = 1.6$, the Newton iteration is not valid for $\cos 1.6$ is a small value. Table 1 shows the iteration procedure, the nearest solution near $x_0 = 1.6$ is $x = \pi$.

Table 1: place caption here

Iteration	$x(\alpha = 1)$	$x(\alpha = 2)$	$x(\alpha) = 0.97$
0	1.6	1.6	1.6
1	2.57	2.09	2.60
2	2.96	2.48	2.98
3	3.12	2.78	3.12
4	3.14	2.99	3.14

we also obtained some iteration formulae by Lagrange multiplier [14]

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f''(x_n)f^2(x_n)}{2f'^3(x_n)} \tag{1.10}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f''(x_n)f^2(x_n)}{2f'^3(x_n)} + \frac{[f'''(x_n)f'^3(x_n) - 3f''^2(x_n)f'^2(x_n)]f^3(x_n)}{2f'^7(x_n)} \tag{1.11}$$

If equation (1.1) is replaced by a differential equation, then a correctional functional similar to equation (1.3) can be established.

1.1 Stationary conditions

The problem of optimization is ubiquitous in nature. The simplest problem of the calculus of variation [12] is to determine a function $y = f(x)$ for which the value of a given functional

$$J = \int_{x_1}^{x_2} F(y, y'; x)dx + g_1(x)y|_{x=x_1} - g_2(x)y|_{x=x_2} \tag{1.12}$$

is a maximum or a minimum.

The extremum condition (stationary condition) of the functional equation (1.12) requires that

$$\begin{aligned}
\delta J &= \delta \int_{x_1}^{x_2} F(y, y'; x)dx + g_1\delta y|_{x=x_1} - g_2\delta y|_{x=x_2} \\
&= \int_{x_1}^{x_2} \delta F(y, y'; x)dx + g_1\delta y|_{x=x_1} - g_2\delta y|_{x=x_2} \\
&= \int_{x_1}^{x_2} \{\frac{dF}{dy}\delta y + \frac{dF}{dy'}\delta y'\}dx + g_1\delta y|_{x=x_1} - g_2\delta y|_{x=x_2} \\
&= \int_{x_1}^{x_2} \{\frac{dF}{dy}\delta y + \frac{dF}{dy'}\frac{d}{dx}(\delta y)\}dx + g_1\delta y|_{x=x_1} - g_2\delta y|_{x=x_2}
\end{aligned}$$

$$= \int_{x_1}^{x_2} \{[\frac{dF}{dy} - \frac{d}{dx}(\frac{dF}{dy'})]\delta y + \frac{d}{dx}(\frac{dF}{dy'}\delta y)\}dx + g_1\delta y|_{x=x_1} - g_2\delta y|_{x=x_2}$$

$$= \int_{x_1}^{x_2} \{[\frac{dF}{dy} - \frac{d}{dx}(\frac{dF}{dy'})]\delta y\}dx + [\frac{dF}{dy'}\delta y]_{x_1}^{x_2} + g_1\delta y|_{x=x_1} - g_2\delta y|_{x=x_2}$$

For arbitrary δy, from the above relation, we have

$$\frac{dF}{dy} - \frac{d}{dx}(\frac{dF}{dy'}) = 0, \tag{1.13}$$

and the boundary conditions

$$\frac{dF}{dy'}(x_1 - g_1(x_1) = 0 and \frac{dF}{dy'}(x_2) - g_2(x_2) = 0. \tag{1.14}$$

Equation (1.13) is called Euler-Lagrange's differential equation, or Euler's equation, and equation (1.14) is known as the natural boundary conditions.

1.2 Restricted Variation

To illustrate how restricted variation works in variational iteration method, we consider a simple algebraic equation

$$x^2 - 3x + 2 = 0. \tag{1.15}$$

We re-write equation (1.15) in the form

$$\tilde{x}.x - 3x + 2 = 0, \tag{1.16}$$

where$\tilde{x}$ is called restricted variable, the value of $\tilde{x}$ is assumed to be known (initial guess). Solving x from equation (1.16) leads to the result

$$x = \frac{2}{3 - \tilde{x}} \tag{1.17}$$

or an iteration form

$$x_{n+1} = \frac{2}{3 - x_n} \tag{1.18}$$

This method is often very efficient for good prediction, see Table 2.

In variational iteration method, initial guess is always chosen with a possible unknown parameter, one iteration leads to highly accurate solutions. We illustrate the effectiveness of free parameter in the above example example. Introducing a free parameter in the initial prediction:

$$x_0 = 0.5 + b, \tag{1.19}$$

where b is a small parameter to be further determined, and substituted equation (1.19) into equation (1.18), we have

$$x_1 = \frac{2}{3 - 0.5 - b} = \frac{2}{2.5 - b} = \frac{2}{2.5(1 - \frac{1}{2.5}b} = 0.8 + 0.32b + 0(b^2), \tag{1.20}$$

To identify the value b we set

$$x_0 = x_1 \tag{1.21}$$

or

$$0.8 + 0.32b = 0.5 + b, \tag{1.22}$$

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From the above relation, we can immediately identify

$$b = 0.4412. \tag{1.23}$$

So the updated approximate root is $x_1 = 0.9412$ Now we consider restricted variation in a variational functional. Consider temperature distribution in convective straight fins with temperature-dependent thermal conductivity, the dimensionless governing equation is [[15]]

$$\frac{d}{dx}[(1+\beta\theta)\frac{d\theta}{dx}] - \psi^2\theta = 0, \theta'(0) = 0, \theta(1) = 1, \tag{1.24}$$

where θ is dimensionless temperature,β and ψ are constants.

Use the concept of restricted variation, an approximate variational functional can be established:

$$J(\theta) = \int_0^1 \{(1+\beta\tilde{\theta})(\frac{d\theta}{dx})^2 + \psi^2\theta^2\}dx, \tag{1.25}$$

where $\tilde{\theta}$ is a restricted variation, i.e., $\delta\tilde{\theta} = 0$. we write equation (1.24) in an iteration form

$$J(\theta_{n+1}) = \int_0^1 \{(1+\beta\theta_n)(\frac{d\theta_{n+1}}{dx})^2 + \psi^2\theta_{n+1}^2\}dx. \tag{1.26}$$

We begin with an initial guess satisfying the boundary condition $\theta_0'(0) = 0$ and $\theta_0 = 1$:

$$\theta_0 = 1 - a + ax^2, \tag{1.27}$$

where a is free parameter. We apply Ritz method to solving θ_1, the trial-function for θ_1 is assumed to have the form

$$\theta_1 = 1 - b + bx^2, \tag{1.28}$$

where the unknown parameter b to be further determined. Substituting equation (1.27) into equation (1.25) yields

$$\left.\begin{aligned} J &= \int_0^1 \{4(1+\beta(1-a+ax^2))b^2x^2 + \psi^2(1-b+bx^2)^2\}dx \\ &= \frac{4}{3}(1+\beta(1-a))b^2 + \frac{4}{5}ab^2 + \psi^2(1-b)^2 + \frac{2}{3}b(1-b)\psi^2 + \frac{1}{5}b^2\psi^2. \end{aligned}\right\} \tag{1.29}$$

Minimizing the functional, equation (1.25), with respect to θ is approximately equivalent to minimizing the above function, equation (1.28), with respect to b:

$$\frac{dJ}{db} = \frac{3}{8}(1+\beta(1-a)b + \frac{8}{5}ab - 2\psi^2(1-b) + \frac{2}{3}(1-2b)\psi^2 + \frac{2}{5}b\psi^2 = 0. \tag{1.30}$$

we set $\theta_0 = \theta_1$, then we can identify

$$a = \frac{-(20+20\beta+8\psi^2) + \sqrt{(20+20\beta+8\psi^2)^2 + 40\psi^2(12-20\beta)}}{24-40\beta} \tag{1.31}$$

The variational iteration method was first proposed by He [17] used to obtain an approximate analytical solutions for nonlinear problems. To clarify the idea of the proposed method for solution of the large deformation of string with large amplitudes, the basic concept of Variational Iteration Method is firstly treated. We consider the following general differential equation,

$$Lu + Nu = g(t) \tag{1.32}$$

where, L is a linear operator, and N a nonlinear operator, g(t) an in homogeneous or forcing term. According to the variational iteration method, we can construct a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(Lu_n(\tau) + N\tilde{u}(\tau) - g(\tau))d\tau \tag{1.33}$$

Where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, the subscript n denotes the nth approximation, $\tilde{u}_n$ is considered as a restricted variation, i.e. $\delta\tilde{u}_n = 0$.

For linear problems, its exact solution can be obtained by only one iteration step due to the fact that the Lagrange multiplier can be exactly identified. In this method, the problems are initially approximated with possible unknowns and it can be applied in non-linear problems without linearization or small parameters. The approximate solutions obtained by the proposed method rapidly converge to the exact solution.

Considering the following example.

The Linear differential equation of first order

$$u' + a(t)u = b(t), u(0) = c. \tag{1.34}$$

Its correction functional can be written in the form

$$u_{n+1}(t) + \int_0^t \lambda\{\frac{du_n}{ds} + a(s)u_n(s) - b(s)\}ds. \tag{1.35}$$

Making the above correct functional stationary with respect to u_n,noticing that $\delta u_n(0) = 0$.

$$\left.\begin{aligned} \delta u_{n+1}(t) &= \delta u_n(t) + \delta\int_0^t \lambda\{\frac{du_n}{ds} + a(s)u_n(s) - b(s)\}ds \\ &= \delta u_n(t) + \lambda\delta u_n(s)|_{s=t} + \int_0^t \{-\frac{\delta\lambda}{\delta s} + a(s)\lambda\}\delta u_n(s)ds = 0. \end{aligned}\right\} \tag{1.36}$$

Thus, we obtain the Euler-Lagrange equation

$$-\frac{\delta\lambda(t,s)}{\delta s} + a(s)\lambda(t,s) = 0, \tag{1.37}$$

and the natural boundary condition

$$1 + \lambda(t,t) = 0. \tag{1.38}$$

We, therefore, identify the Lagrange multiplier in the form

$$\lambda(t,s) = -\exp\{\int_0^s a(\xi)d\xi - \int_0^t a(\xi)d\xi\} \tag{1.39}$$

Substituting the identified Lagrange multiplier into equation (1.2) results in the following iteration formulation:

$$u_{n+1}(t) = u_n(t) - \int_0^t \exp\{\int_0^s a(\xi)d\xi - \int_0^t a(\xi)d\xi\}\{\frac{du_n}{ds} + a(s)u_n(s) - b(s)\}ds \tag{1.40}$$

If we begin with $u_0 = c\exp\{-\int_0^t a(\xi)d\xi\}$,the solution the homogeneous equation, u'+a(t)u=0 with initial condition u(0)=c, then

$$\left.\begin{aligned} u_1(t) &= c\exp\{-\int_0^t a(\xi)d\xi\} - \int_0^t b(s)\exp\{\int_0^s a(\xi)d\xi - \int_0^t a(\xi)d\xi\}ds \\ &= c\exp\{-\int_0^t a(\xi)d\xi\} + \exp\{\int_0^t a(s)ds\}\int_0^t b(s)\exp\{a(\xi)d\xi\}ds. \end{aligned}\right\} \tag{1.41}$$

which is the exact explicit solution. From the solution all properties can be quickly and effortlessly deduced. Now consider the linear oscillator with a forced term:

$$u''(t) + \omega^2 u(t) = f(u, u', u''). \tag{1.42}$$

Its correction functional can be written in the form

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\{u_n''(s) + \omega^2 u_n(s) - \tilde{f}(u_n, u_n', u_n'')\}ds. \tag{1.43}$$

Calculating variation with respect to u_n, noticing that $\delta u_n(0) = 0$, yields

$$\left.\begin{aligned} \delta u_{n+1}(t) &= \delta u_n(t) + \delta \int_0^t \lambda\{u_n''(s) + \omega^2 u_n(s) - f(s)\}ds \\ &= \delta u_n(t) + \lambda(s)\delta u_n'(s)|_{s=t} - \frac{\delta\lambda}{\delta s}\delta u_n(s)|_{s=t} + \int_0^t \{\frac{\delta^2\lambda}{\delta s^2} + \omega^2\lambda(s)\}\delta u_n(s)ds \\ &= (1 - \frac{\delta\lambda}{\delta s})\delta u_n(s)|s = t + \lambda(s)\delta\delta u_n'(s)|_{s=t} + \int_0^t \{\frac{\delta^2\lambda}{\delta s^2} + \omega^2\lambda(s)\}\delta u_n(s)ds = 0. \end{aligned}\right\} \tag{1.44}$$

We, therefore, have the following stationary conditions:

$$\frac{\delta^2\lambda(t,s)}{\delta s^2} + \omega^2\lambda(t,s) = 0, \tag{1.45}$$

$$1 - \frac{\delta\lambda(t,s)}{\delta s^2}|_{t=s} = 0, \tag{1.46}$$

$$\lambda(t,s)|_{t=s} = 0. \tag{1.47}$$

The Lagrange multiplier, therefore, can be readily identified

$$\lambda = \frac{1}{\omega} sin\omega(s - t). \tag{1.48}$$

A a result, we obtain the following iteration formula:

$$u_{n+1}(t) = u_n(t) + \frac{1}{\omega}\int_0^t sin\omega(s-t)\{u_n''(s) + \omega^2 u_n(s) - f(u_n, u_n', u_n'')\}ds. \tag{1.49}$$

If we use its complementary solution $u_0(t) = C_1 cos\omega t + C_2 sin\omega t$ with suitable constants C_1 and C_2 as an initial approximation, by the iteration formula equation (1.18), we have

$$u_1(t) = C_1 cos\omega t + C_2 sin\omega t - \frac{1}{\omega}\int_0^t f(s) sin\omega(s-t)ds, \tag{1.50}$$

which is the general solution of equation (1.11).

For nonlinear equation, the Lagrange multiplier is difficult to be identify. To overcome the difficulty, we apply restricted variations to nonlinear terms. In order to best illustrate the restricted variations, we consider the following linear equations:

$$u''(t) + \omega^2 u(t) = 0. \tag{1.51}$$

We write the correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\{u_n''(s) + \omega^2 \tilde{u}_n(s)\}ds. \tag{1.52}$$

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Herein $\tilde{u}_n$ is considered as a restricted variation, i.e., $\delta\tilde{u}_n = 0$. Under this condition, its stationary conditions of the above correction functional equation (1.21) can be expressed as follows:

$$\frac{\delta^2\lambda(t,s)}{\delta s^2} = 0, \tag{1.53}$$

$$1 - \frac{\delta\lambda(t,s)}{\delta s}|_{t=s} = 0, \tag{1.54}$$

$$\lambda(t,s)|_{t=s} = 0. \tag{1.55}$$

The Lagrange multiplier, therefore, can be easily identified as

$$\lambda = s - t, \tag{1.56}$$

leading to the following iteration formula:

$$u_{n+1}(t) = u_n(t) + \int_0^t (s-t)\{u_n''(s) + \omega^2 u_n(s)\}ds. \tag{1.57}$$

If, for example, the initial conditions are u(0)=1 and u'(0)=0, we begin with $u_0 = u(0) = 1$, by the above iteration formula equation (1.26) we have the following approximate solutions:

$$u_1(t) = 1 - \frac{1}{2!}\omega^2 t^2, \tag{1.58}$$

$$u_2(t) = 1 - \frac{1}{2!}\omega^2 t^2 + \frac{1}{4!}\omega^4 t^4, \tag{1.59}$$

and

$$u_n(t) = 1 - \frac{1}{2!}\omega^2 t^2 + ... + (-1)^n \frac{1}{(2n)!}\omega^{2n} t^{2n}. \tag{1.60}$$

From the above solution procedure, we can see clearly that the approximate solutions converge to its exact solution $cos\omega t$ relatively slowly due to the approximate identification of the multiplier. It should be specially pointed out that the more accurate the identification of the multiplier, the more fast the approximations converge to its exact solution.

Now consider the following nonlinear equation:

$$u' + a(t)u = b(t) + N(u), u(0) = c. \tag{1.61}$$

Here N is a nonlinear function of u. Following the above notation, we may set

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\{\frac{du_n}{ds} + a(s)u_n(s) - b(s) - N(\tilde{u_n})\}ds, \tag{1.62}$$

where $\tilde{u_n}$ is a restricted variation, i.e., $\delta\tilde{u_n} = 0$.

Proceeding along the lines indicated above, we obtain the following iteration formulation:

$$u_{n+1}(t) = u_n(t) - \int_0^t \exp\{\int_0^s a(\xi)d\xi - \int_0^t a(\xi)d\xi\}\{\frac{du_n}{ds} + a(s)u_n(s) - b(s) - N(u_n)\}ds. \tag{1.63}$$

Consider the following second-order nonlinear equation

$$u'' + a(t)u' + b(t)u + N(u) = 0, u(0) = c_1, u'(0) = c_2. \tag{1.64}$$

Its correction functional can be written in the form

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\{u_n''(s) + a(s)u_n'(s) + b(s)u_n(s) + N(\tilde{u^n}(s))\}ds. \tag{1.65}$$

Making the above correct functional stationary with respect to u_n, noticing that $\delta u_n(0) = 0$,

$$\left.\begin{aligned} \delta u_{n+1}(t) &= \delta u_n(t) + \int_0^t \lambda\{u_n''(s) + au_n'(s) + bu_n(s) + N(\tilde{u_n(s)})\}ds \\ &= \delta u_n(t) + \lambda\delta u_n'(s)|_{s=t} + a\lambda(s)\delta u_n(s)|_{s=t} \\ &+ \int_0^t \{\frac{\delta^2\lambda}{\delta s^2} - \frac{\delta}{\delta s}(a\lambda) + b(s)\lambda(s)\}\delta u_n(s)ds = 0, \end{aligned}\right\} \tag{1.66}$$

we have the following stationary conditions:

$$\frac{\delta^2\lambda}{\delta s^2} - \frac{\delta}{\delta s}(a\lambda) + b(s)\lambda(t,s) = 0, \tag{1.67}$$

$$1 - \frac{\delta\lambda}{\delta s}(t,t) + a(t)\lambda(t,t) = 0, \tag{1.68}$$

$$\lambda(t,t) = 0. \tag{1.69}$$

From the above relations, equation (1.40)- (1.20), we can identify the Lagrange multiplier as follows:

$$\lambda = \exp\{\int_o^s a(\xi)d\xi\}[u_1(s)u_2(t) - u_2(s)u_1(t)]. \tag{1.70}$$

If w_1 and w_2 are chosen to be a fundamental set of solutions specified by the initial conditions

$$w_1(0) = 1, w_1'(0) = 0, w_2(0) = 0, w_2'(0) = 1 \tag{1.71}$$

To first approximation, we have

$$u_1(t) = c_1w_1 + c_2w_2 + \int_0^t N(u_0(s))\lambda(t,s)ds. \tag{1.72}$$

Consider the following in homogeneous equation:

$$u'' + a(t)u' + b(t)u = f(u), \tag{1.73}$$

where the solution is specified by values at two points, say

$$u(0) = c_1, u(1) = c_2 \tag{1.74}$$

We shall let $u_{(1)}$ and $u_{(2)}$ to the two principal solutions; that is

$$u_{(1)}(0) = 1, u_{(1)}'(0) = 0, \tag{1.75}$$

$$u_{(2)}(0), u_{(2)}'(0) = 1, \tag{1.76}$$

and write

$$u = c_1u_{(1)} + c_2u_{(2)}. \tag{1.77}$$

Its correction functional can be written in the form

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\{u_n''(s) + au_n'(s) + bu_n(s) - \tilde{f}_n\}ds. \tag{1.78}$$

Making the above correct functional stationary with respect to u_n, noticing that $\delta u_n(0) = 0$,

$$\delta u_{n+1}(t) = \delta u_n(t) + \delta\int_0^t \lambda\{u_n''(s) + au_n'(s) + bu_n(s) - \tilde{f}_n\}ds$$

$$= \delta u_n(t) + \lambda(s)\delta u_n'(s)|_{s=t} - \lambda'(s)\delta u_n(s)|_{s=t} + a\lambda(s)\delta u_n(s)|_{s=t}$$
$$+ \int_0^t \{\frac{d^2\lambda}{ds^2} - \frac{d}{ds}(a\lambda) + b(s)\lambda(s)\}\delta u_n(s)ds = 0,$$

we have the following stationary conditions:

$$\{\delta u_n(s) : \frac{d^2\lambda}{ds^2} - \frac{d}{ds}(a\lambda) + b(s)\lambda(t,s) = 0, \delta u_n(t) : 1 - \frac{d\lambda}{ds} + a(s)\lambda(s,s) = 0, \delta u_n'(t) : \lambda(s,s) = 0. \quad (1.79)$$

The Lagrange multiplier can be identified in the form

$$\lambda = -\exp\{\int_0^s a(\xi)d\xi\}[u_{(1)}(s)u_2(t) - u_2(s)u_1(t)]. \quad (1.80)$$

So we obtain the following iteration formulation:

$$u_{n+1}(t) = u_n(t) - \int_0^t \exp\{\int_0^s a(\xi)d\xi\}[u_{(1)}(s)u_2(t) - u_2(s)u_1(t)]$$
$$\{u_n''(s) + au_n'(s) + bu_n(s) - f_n\}ds. \quad (1.81)$$

2 APPLICATION OF THE PROPOSED METHOD

Example 2.1. *In this example, Mathematical Pendulum that was studied by He [[18]] is considered. The differential equation of motion of the undamped mathematical pendulum is given by,*

$$\ddot{y} + \omega^2 siny = 0 \quad (2.1)$$

The initial conditions for this problem are as follows:

$$y(0) = A \quad (2.2)$$
$$\dot{y}(0) = 0 \quad (2.3)$$

The sin y term in equation (1.3) is a nonlinear term and it can be expanded as

$$siny \approx y - \frac{1}{6}y^3 \quad (2.4)$$

Substituting equation (1.5) into equation (1.3) gives

$$\ddot{y} + \omega^2 y - \frac{\omega^2}{6}y^3 = 0 \quad (2.5)$$

The Lagrange multiplier of this problem is

$$\lambda = \frac{1}{\omega}sin[\omega(\tau - t)] \quad (2.6)$$

Hence the iteration formula is

$$y_{n+1}(t) = y_n(t) + \frac{1}{\omega}\int_0^t sin[\omega(\tau - t)][y^n(\tau) + \omega^2 y(\tau) - \frac{\omega^2}{6}y^3(\tau)]d\tau \quad (2.7)$$

The complementary solution of this problem that is used as an initial approximation is given by

$$y_0(t) = Acos(\alpha\omega t) \quad (2.8)$$

where α is an unknown constant.

Substituting the initial approximation into equation (1.6), the following residual is obtained

$$R_0(t) \approx \ddot{y} + \omega^2 y - \frac{\omega^2}{6}y^3 = A(1 - \frac{1}{8}A^2 - \alpha^2)\omega^2 cos(\alpha\omega t) - \frac{1}{24}A^3\omega^2 cos(3\alpha\omega t) \tag{2.9}$$

The coefficient of the $cos(\alpha\omega t)$ term is set to zero in order to eliminate the secular term which may occur in the next iteration. Doing so, the expression of α is found as follows

$$\alpha = \sqrt{1 - \frac{A^2}{8}} \tag{2.10}$$

Hence,

$$y_1(t) = Acos\alpha\omega t - \frac{A^3}{24(9\alpha^2 - 1)\omega^2}(cos3\alpha\omega t - cos\omega t) \tag{2.11}$$

where α defined in equation (1.12). The period can be expressed as follows

$$T = \frac{2\pi}{\omega\sqrt{1 - \frac{1}{8}A^2}} \tag{2.12}$$

If $A = \frac{\pi}{2}$, then $T = 1.20T_0$. On the hand He's [[18], [19]] approximation gives $T = 1.17T_0$, while the exact period is $T_{ex} = 1.16T_0$, where $T_0 = \frac{2\pi}{\omega}$.

Example 2.2. *In this example, the problem that was studied by Nayfeh and Mook [[20]] is considered.*

The differential equation of motion is given by,

$$\ddot{u} + \omega^2 u + \varepsilon u^2 \ddot{u} = 0 \tag{2.13}$$

The initial conditions for this problem are as follows:

$$y(0) = A \tag{2.14}$$

$$\ddot{y}(0) = 0 \tag{2.15}$$

The Lagrange multiplier of this problem is

$$\lambda = \frac{1}{\omega}sin[\omega(\tau - t)] \tag{2.16}$$

The iteration formula is given by

$$u_{n+1}(t) = u_n(t) + \frac{1}{\omega}\int_0^t sin[\omega(\tau - t)][u^n.(\tau) + \omega^2 u(\tau) + \varepsilon u^2(\tau)u^n(\tau)]d\tau \tag{2.17}$$

The complementary solution of this problem that is used as an initial approximation is given by

$$u_0(t) = Acos(\alpha\omega t) \tag{2.18}$$

where α is an unknown constant.

Substituting the initial approximation given by equation (1.20), the following residual is obtained as follows

$$R_0(t) \approx \ddot{u} + \omega^2 u + \varepsilon u^2 \ddot{u} = A\omega^2(1 - \alpha^2 - \frac{3}{4}A^2\varepsilon)cos(\alpha\omega t) - \frac{1}{4}\varepsilon A^3\alpha^2\omega^2 cos(3\alpha\omega t) \tag{2.19}$$

The coefficient of the cos($\alpha\omega t$) term is set to zero in order to eliminate the secular term which may occur in the next iteration. Doing so, the expression for α is obtained as follows

$$\alpha = \frac{2}{\sqrt{4+3\varepsilon A^2}} \tag{2.20}$$

Hence,

$$y_1(t) = Acos\alpha\omega t + \frac{\varepsilon A^3\alpha^2}{4(9^2-1)}(cos\omega t - cos3\alpha\omega t) \tag{2.21}$$

where α defined in equation (1.23). The new frequency is defined as follows

$$\omega_1 = \alpha\omega \Rightarrow \omega_1 = \frac{2}{\sqrt{4+3\varepsilon A^2}}\omega \tag{2.22}$$

The frequency that is obtained by Nayfeh and Mook [[20]] using the perturbation method is

$$\omega_1 = \omega(1 - \frac{3}{8}\varepsilon A^2) \tag{2.23}$$

Note that equation (1.25) is valid only for smallε values. However, the frequency expression given by equation (1.24) is valid for all ε values and takes the following form for small ε values

$$\omega_1 \approx 1 - \frac{3}{8}\varepsilon A^2 + \frac{27}{128}\varepsilon^2 A^4 + ... \tag{2.24}$$

Example 2.3. *In this example, the Duffing-harmonic oscillator that was studied by Lim and Wu [21] and Mickens [22] is considered.*

The differential equation of motion is given by,

$$\frac{d^2y}{dt^2} + \frac{y^3}{1+y^2} \tag{2.25}$$

The initial conditions for this problem are as follows:

$$y(0) = A \tag{2.26}$$

$$\dot{y}(0) = 0 \tag{2.27}$$

with initial conditions y(0)=A and y'(0)=0.

The following form of equation (1.27) is going to be studied in this example

$$(1+y^2)\frac{d^2y}{dt^2} + y^3 = 0 \tag{2.28}$$

He's technique is going to be used to overcome seculer terms that appear in the iterations. The initial approximation is,

$$y_0(t) = Acos(\alpha t) \tag{2.29}$$

where α is an unknown constant.

Substituting the initial approximation into equation (1.30), the following residual is obtained

$$R_0(t) \approx (1+y^2)\ddot{y} + y^3 = (\frac{3}{4}A^2 - \alpha^2 - \frac{3}{4}A^2\alpha^2)cos(\alpha t) + \frac{A^3}{4}(1-\alpha^2)cos(3\alpha t) \tag{2.30}$$

In order to discard the seculer terms, the coefficient of cos(αt) is set to zero which gives the expression of α as follows

$$\alpha = \sqrt{\frac{\frac{3}{4}A^2}{1+\frac{3}{4}A^2}} \tag{2.31}$$

Hence the new frequency is defined as follows

$$\omega = \sqrt{\frac{\frac{3}{4}A^2}{1+\frac{3}{4}A^2}} \tag{2.32}$$

which is the same with the one found by Lim and Wu [21] and Mickens [22].

The iteration formula is given by

$$u_1(t) = u_0(t) + \int_0^t (\tau - t)[\frac{A^3}{4}(1-\alpha^2)cos(3\alpha\tau)]d\tau \tag{2.33}$$

Hence,

$$y_1(t) = cos\omega t + \frac{A}{27}(cos3\omega t - 1) \tag{2.34}$$

where ω defined in equation (2.38).

Example 2.4. *(A ball-bearing oscillator). In this example we consider the motion of a ball-bearing oscillating in a smooth tube that is bent into a curve such that the restoring force depends upon the cube of the displacement. The governing equation, which can be readily obtained, is*

$$u'' + \varepsilon u^3 = 0, \tag{2.35}$$

with initial conditions u(0)=A,u'(0)=0.

The iteration formula for the system can be constructed as follows:

$$u_{n+1}(t) = u_n(t) + \frac{1}{\omega}\int_0^t \sin\omega(s-t)\{u_n''(s) + \varepsilon u_n^3(s)\}ds, \tag{2.36}$$

where ω is the angular frequency of the system.

We begin with $u_0(t) = Acos\omega t$, by formula equation (2.38), we have

$$u_1(t) = A\cos\omega t - \frac{1}{32}A^3(\cos 3\omega t - \cos\omega t), \tag{2.37}$$

where the angular frequency ω is identified with the physical understanding that no secular terms should be appeared in $u_1(t)$, which leads to

$$\omega = \frac{\sqrt{3}}{2}\varepsilon^{\frac{1}{2}}A. \tag{2.38}$$

The approximate period can be expressed in the form

$$T = \frac{4\sqrt{3}\pi}{3\varepsilon^{\frac{1}{2}}A}, \tag{2.39}$$

while the exact period can be readily obtained, which reads

$$T_{ex} = 2\sqrt{2}\int_0^A \frac{dx}{\sqrt{\int_x^A g(u)du}}, \tag{2.40}$$

where $g(u) = \varepsilon u^3$. For any $A > 0$ and $\varepsilon > 0 >$, equation (2.42) can be approximated as

$$T_{app} = \frac{\pi^{\frac{3}{2}}A^{\frac{1}{2}}}{\sqrt{\int^{\frac{\pi}{2}}{}_0 g(A\cos x)\cos x dx}} = \frac{\pi^{\frac{3}{2}}A^{\frac{1}{2}}}{\sqrt{\int_0^{\frac{\pi}{2}} \varepsilon A^3\cos^4 x dx}} = \frac{4\pi}{\sqrt{3}\varepsilon^{\frac{1}{2}}A}. \tag{2.41}$$

So the results obtained by the present theory are valid for $A > 0$ and $\varepsilon > 0$.

Example 2.5. *We consider an oscillator, which is governed by*

$$u'' + \omega^2 u + 4\varepsilon u^2 u'' + 4\varepsilon u u'^2 = 0, u(0) = A, u'(0) = 0. \tag{2.42}$$

Supposing system equation (2.44) has the angular frequency Ω, we obtain the following iteration formula:

$$\left.\begin{aligned} u_{n+1}(t) = u_n(t) + \frac{1}{\Omega}\int_0^t \sin\Omega(s-t)\{u_n''(s) + \omega^2 u_n(s)\}ds \\ + \frac{1}{\Omega}\int_0^t \sin\Omega(s-t)\{4\varepsilon u_n^2(s)u''(s) + 4\varepsilon u_n(s)u_n'^2(s)\}ds. \end{aligned}\right\} \tag{2.43}$$

We begin with the initial approximation $u_0(t) = A\cos\Omega t$, by the above iteration formula, we have

$$u_1(t) = A\cos\Omega t - \frac{\varepsilon A^3}{4}(\cos 3\Omega t - \cos\Omega t), \tag{2.44}$$

where the angular frequency is identified with the understanding that no secular terms occur, which leads to

$$\Omega = \frac{\omega}{\sqrt{1 + 2\varepsilon A^2}}. \tag{2.45}$$

The period is

$$T = T_0\sqrt{1 + 2\varepsilon A^2}, T_0 = \frac{2\pi}{\omega}, \tag{2.46}$$

while the exact periods reads

$$T_{ex} = \frac{2}{\pi}T_0\sqrt{1 + 4\varepsilon A^2}\int_0^{\frac{\pi}{2}} \sqrt{1 - k\sin^2 t}dt, \tag{2.47}$$

where $k = 4\varepsilon A^2/(1 + 4\varepsilon A^2)$.

It is obvious the obtained result equation (2.48) is valid for all $\varepsilon > 0$. Even in case $\varepsilon \to \infty$, we have

$$lim_{\varepsilon\to\infty}\frac{T_{ex}}{T} = 0.9003$$

Example 2.6. *(Schrodinger equation). The Schrodinger equation can be written in the form*

$$Lf = -f'' + q(x)f = k^2 f. \tag{2.48}$$

In case q(x)=0, we have two general solutions: e^{ikx} and e^{-ikx}. We assume that $q(x) \to 0$ when $|x| \to \infty$

In case $x \to +\infty$, the correction functional can be constructed as follows:

$$f_{n+1}(x) = f_n(x) + \int_x^{+\infty} \lambda\{f_n''(\xi) - q(\xi)\tilde{f}_n(\xi)\}d\xi. \tag{2.49}$$

where $\tilde{f}_n$ is considered as a restricted variation.

Making equation (2.51) stationary, we obtain the following stationary conditions

$$\lambda''(\xi) + k^2 f(\xi) = 0, \lambda(\xi)|_{\xi=x} = 0, 1 + \lambda'(\xi)|_{\xi=x} = 0. \tag{2.50}$$

The Lagrange multiplier can be identified as

$$\lambda = \frac{1}{k}\sin k(x - \xi). \tag{2.51}$$

We obtain the following iteration formulation:

$$f_{n+1}(x) = f_n(x) + \int_x^{+\infty} \frac{\sin k(x-\xi)}{k}\{f_n''(\xi) - q(\xi)\tilde{f}_n(\xi)\}d\xi. \tag{2.52}$$

We begin with

$$f_0(x) = f + (x) = e^{ikx}. \tag{2.53}$$

By (11), we have

$$f_1(x) = e^{ikx} - \int_x^{+\infty} \frac{\sin k(x-\xi)}{k} q(\xi) - f + (\xi)d\xi. \tag{2.54}$$

Now we consider another case when $x \to -\infty$. Under such a case, we construct a correction functional in the form

$$f_{n+1}(x) = f_n(x) + \int_{-\infty}^{x} \lambda\{f_n''(\xi) - q(\xi)\tilde{f}_n(\xi) + k^2 f_n(\xi)\}d\xi. \tag{2.55}$$

Its stationary conditions are

$$\lambda''(\xi) + k^2 f(\xi) = 0, \lambda(\xi)|_{\xi=x} = 0, 1 - \lambda'(\xi)|_{\xi=x} = 0. \tag{2.56}$$

The multiplier can be identified as

$$\lambda = -\frac{1}{k}\sin k(x-\xi). \tag{2.57}$$

so we obtain the following iteration formulation:

$$f_{n+1}(x) = f_n(x) - \int_x^{+\infty} \frac{\sin k(x-\xi}{k}\{f_n''(\xi) - q(\xi)f_n(\xi) + k^2 f_n(\xi)\}d\xi. \tag{2.58}$$

We begin with

$$f_0(x) = f - (x) = e^{-ikx}. \tag{2.59}$$

By equation (2.60), we have

$$f_1(x) = e^{ikx} + \int_x^{+\infty} \frac{\sin k(x-\xi}{k} q(\xi) f - (\xi)d\xi. \tag{2.60}$$

Equation (2.56) and equation (2.60) are actually Jost solutions.

3 CONCLUSIONS

In this paper, the main purpose was to illustrate the application of VIM in solving nonlinear oscillator systems. The variational iteration algorithm leads to analytical solutions in the present study. Additionally, the procedure presented in this paper can be simply extended to solve more complex vibration problems; such as aeroelasticity, random vibrations etc. The Variational Iteration method can be easily comprehended with only a basic knowledge of advanced calculus.

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Competing financial interests

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