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G-cone metric Spaces over Banach Algebras and Some Fixed Point Results

Adewale, O. K.[1*] And Osawaru E. K. [2]

1 *, Department of Mathematics, University of Lagos, Akoka, Nigeria.
2. Department of Mathematics, University of Benin, Nigeria.
*Corresponding author: adewalekayode2@yahoo.com

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Abstract

In this paper, the notion of a G-cone metric space over Banach algebras and the generalized contractive mapping defined on G-cone metric space over Banach algebras are introduced. Some new fixed point results for the maps are proved without the assumption of normality. The results are significant extensions of fixed point results of maps on G-cone metric and metric spaces in literature.

Keywords: Banach algebras, fixed point, G-cone metric spaces, Spectral radius.
MSC2010: 47H10, 54H25.

1 Introduction

2-metric space was introduced by Gahler [1] in 1963. Gahler [2] claimed that the space was a generalization of a metric space, but different authors proved that there is no relationship between these two functions. For instance Ha *et al.* [3] showed that a 2-metric need not be a continuous function of its variables, whereas an ordinary metric is and that there is no easy relationship between results obtained in the two settings. In particular, the contraction mapping theorem in metric spaces and in 2-metric spaces are unrelated.
In 1992, Dhage [4] proposed the notion of a D metric space in an attempt to obtain analogous results to those for metric spaces, but in a more general setting. In a subsequent series of papers (including: Dhage [5], Dhage [4], Dhage [6] and Dhage [7]), Dhage [6] presented topological structures in such spaces together with several fixed point results. These research efforts have been the basis for a substantial number of results by other authors. Unfortunately, in 2006, Mustafa and Sims proved that these attempts are invalid. They later introduced a new structure of generalised metric spaces. This is called G-metric space-a generalisation of the usual metric space (see Mustafa and Sims [8], Mustafa and Sims [9], Mustafa and Sims [10], Mustafa [11]). They also proved fixed point theorems for various mappings in this new structure.

On the other hand, cone metric space was introduced by Huang and Zhang [12] as a generalization of metric space. The distance between any two points in a cone metric space is defined to be a vector in an ordered Banach space E. In Huang and Zhang [12], Olaleru [13] and Olaleru [14], the authors proved that there exists a unique fixed point for contractive mappings in complete cone metric spaces.

However, results in cone metric spaces over Banach spaces are argued by researchers as not significant as there are equivalent metric spaces defined by the nonlinear scalarization function implying that cone metric spaces are special cases of classical metric spaces (see Arandelovic and Keckic [15], Du [16]). After that, some other interesting generalizations were developed. (See Abbas *et al* [17]). For example, in Hao and Shaoyuan [18], the Banach space E was replaced by a Banach algebra A and the concept of cone metric spaces over Banach algebras was studied. The authors of Hao and Shaoyuan [18] also proved some fixed point theorems of generalized Lipschitz mappings with weaker and natural conditions on the Lipschitz constant k. These results are significant as the contraction constant k satisfies the condition that $r(k) \in [0, 1)$ which is weaker than the condition $\|k\| \in [0, 1)$, a requirement in the cone metric result settings ($r(k)$ denotes the spectra radius of the vector k Rudin [19]).

In this paper, the notion of G-cone metric space over Banach algebra is introduced. Some fixed point results both in metric spaces and cone metric space with Banach algebras are generalized with an application.

We give some basic definitions of concepts which serve as background to this work. Throughout this paper, we assume that P is a cone in A with $intP \neq \emptyset$ (θ, the additive identity element of A) and $\preceq$ is the partial ordering with respect to P where A is a real Banach algebra. That is, A is a real Banach space in which an operation of multiplication is defined, satisfying the following properties (for all $x, y, z \in A$, $\alpha \in \mathbb{R}$):

(i) $x(yz) = (xy)z$;

(ii) $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$;

(iii) $\alpha(xy) = (\alpha x)y = x(\alpha y)$;

(iv) $\|xy\| \leq \|x\|\|y\|$

Proposition 1.1 Hao and Shaoyuan [18]
Let A be a Banach algebra with a unit e, and $x \in A$. If the spectral radius, $r(x)$ of x is less than 1, i.e.,

$$r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \|x^n\|^{\frac{1}{n}} < 1$$

then $e - x$ is invertible. In fact,

$$(x - e)^{-1} = \sum_{i=0}^{\infty} x^i.$$

A subset P of A is called a cone if:

(i) P is non-empty closed and $\{\theta, e\} \subset P$;

(ii) $\alpha P + \beta P \subset P$ for all non-negative real numbers α, β;

(iii) $P^2 = PP \subset P$;

(iv) $P \cap \{-P\} = \{\theta\}$.

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For a given cone $P \subset A$, we can define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. $x \prec y$ will stand for $x \preceq y$ and $x \neq y$. While $x \ll y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of P. The cone P is called normal if there is a number $M > 0$ such that for all $x, y \in A$,

$$\theta \preceq x \preceq y \Rightarrow \|x\| \leq M\|y\|$$

The least positive number satisfying the above condition is called the normal constant of P Mishra *et al.* [18].

Remark 1.2 Xu and Radenovic [20].
If $r(x) < 1$, then $\|x^n\| \to 0$ as $n \to \infty$

Definition 1.3 Hao and Shaoyuan [18].
Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}$ a sequence in X. Then
(i) $\{x_n\}$ converges to x if for each $c \in A$ with $\theta \ll c$, there is a natural number N such that $d(x_n, x) \prec c$ for all $n \geq N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.
(ii) $\{x_n\}$ is a Cauchy sequence if for each $c \in A$ with $\theta \ll c$, there is a natural number N such that $d(x_n, x_m) \prec c$ for all $n, m \geq N$.
(iii) (X, d) is a complete cone metric space if every Cauchy sequence is convergent in X.

Lemma 1.4 Xu and Radenovic [20].
Let A be a Banach algebra and k, a vector in A. If $0 \leq r(k) < 1$, then we have

$$r((e-k)^{-1}) < (1 - r(k))^{-1}.$$

Lemma 1.5 Xu and Radenovic [20].
Let A be a Banach algebra and x, y be vectors in A. If x and y commute, then the following holds:
1. $r(xy) \leq r(x)r(y)$;
2. $r(x+y) \leq r(x) + r(y)$;
3. $|r(x) - r(y)| \leq r(x-y)$.

Lemma 1.6 Radenovic and Rhoades [21].
If A is real Banach algebra with a solid cone P and $\{x_n\}$ is a sequence in A. Suppose $\|x_n\| \to 0 (n \to \infty)$ for any $\theta \ll c$. Then $x_n \ll c$ for all $n > N^1, N^1 \in N$.

Lemma 1.7 Xu and Radenovic [20].
If E is a real Banach space with a solid cone P and if $\| x_n \| \to 0$ as $n \to \infty$, then for any $\theta \ll c$, there exists $N \in \mathbb{N}$ such that, for any $n > N$, we have $x_n \ll c$.

Example 1.8 Hao and Shaoyuan [18].
Let A be the Banach space of all continuous real-valued functions $C(K)$ on a compact Hausdorff topological space K, with multiplication defined pointwise. Then A is a Banach algebra, and the constant function $f(t) = 1$ is the unit of A.
Let $P = \{f \in A : f(t) \geq 0 \text{ for all } t \in K\}$. Then $P \subset A$ is a normal cone with a normal constant $M = 1$.
Let $X = C(K)$ with the metric $d : X \times X \to A$ defined by

$$d(f, g) = |f(t) - g(t)|$$

where $t \in K$.
Then (X, d) is a cone metric space over a Banach algebra A.

2 Discussion of Results

G-cone Metric Space over a Banach algebra
In this section, we introduce the notion of G-cone metric space over Banach algebra and prove some fixed point theorems in this new space.

Definition 2.1.
Let X be a non-empty set, A, a Banach algebra and $G : X^3 \to A$ be a function satisfying the following properties:

(i) $G(x, y, z) = \theta \quad$ if and only if $x = y = z$

(ii) $\theta \prec G(x, x, y), \quad \forall x, y \in X$, with $x \neq y$

(iii) $G(x, x, y) \preceq G(x, y, z), \quad \forall x, y, z \in X$, with $z \neq y$

(iv) $G(x, y, z) = G(y, z, x) = G(x, z, y) = ...$ (symmetry).

(v) $G(x, y, z) \preceq G(x, a, a) + G(a, y, z) \quad \forall a, x, y, z \in X$ (rectangle inequality)

Then G is called a G-cone metric over Banach algebra A and the pair (X, G) denotes a G-cone metric space over Banach algebra.

Remark 2.2

(i) If A is a Banach space in Definition 2.1, then (X, G) becomes a G-cone metric space and if in addition $z = y$, then it becomes a cone metric space as in Huang and Zhang [12].

(ii) If $A = \mathbb{R}$ in Definition 2.1, we obtain a G-metric space as in Mustafa and Sims [8] and if in addition, $z = y$ in $G(x, y, z)$, then it becomes a metric space as in Frechet [22].

Some properties of the space are stated as follows:

Definition 2.3
Let (X, G) be a G-cone metric space over Banach algebra. G is said to be symmetric if:

$$G(x, y, y) = G(x, x, y)$$

for all $x, y, z \in X$.

Definition 2.4
A G-cone metric space over Banach algebra A is said to be G-bounded if for any $x, y, z \in X$, there exists $K \succ \theta$ such that $\| G(x, y, z) \| \preceq K$.

Definition 2.5
Let (X, G) be a G-cone metric space over Banach algebra and $\{x_n\}$ a sequence in X, $c \gg \theta$ with $c \in A$. Then
1. $\{x_n\}$ converges to $x \in X$ if and only if $G(x_m, x_n, x)| \ll c$ for all $n, m > N^1, N^1 \in N$.
2. $\{x_n\}$ is Cauchy sequence if and only if $G(x_n, x_m, x_p) \ll c$ for all $n, m > p > N^1, N^1 \in N$.
3. (X, G) is complete G -cone metric space over Banach algebra if every Cauchy sequence converges.

Example 2.6
Let A be the Banach space of all continuous real-valued functions $C(K)$ on a compact Hausdorff topological space K, with multiplication defined pointwise. Then A is a Banach algebra, and the constant function $f(t) = 1$ is the unit of A, where $t \in K$.

Let $P = \{f \in A : f(t) \geq 0 \text{ for all } t \in K\}$. Then $P \subset A$ is a normal cone with a normal constant $M = 1$.
Let $X = C(K)$ with the map $G : X \times X \times X \to A$ defined by

$$G(f, g, h) = \max_{t \in K}\{|f(t) - g(t)|, |g(t) - h(t)|, |h(t) - f(t)|\}.$$

Then (X, G) is a G-cone metric space over Banach algebra A.

Some fixed point results of maps defined on the space are stated and proved. The novelty/significance of the results obtained are also discussed.

Theorem 2.7
Let (X, G) be a complete G-cone metric space over a Banach algebra A and P be a non-normal cone. Suppose that the mapping $T : X \to X$ satisfies the generalized contraction condition

$$G(Tx, Ty, Tz) \preceq kG(x, y, z), \forall x, y \in X$$

where $0 < r(k) < 1, k \in P, x, y, z \in X$. Then T has a unique fixed point in X.

Proof:
Choose $x_0 \in X$ and set $x_n = T^n x_0$, $n \geq 1$. We have

$$G(x_n, x_{n+1}, x_{n+1}) = G(Tx_{n-1}, Tx_n, Tx_n) \preceq kG(x_{n-1}, x_n, x_n) \preceq \cdots \preceq k^n G(x_0, x_1, x_1)$$

Thus, for $m > n$,

$$\begin{aligned} G(x_n, x_m, x_m) &\preceq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_{n+3}, x_{n+3}) \\ &\quad + \cdots + G(x_{m-1}, x_m, x_m) \end{aligned}$$

From Equations (1)and (2), we have

$$\begin{aligned} G(x_n, x_m, x_m) &\preceq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + G(x_{n+2}, x_{n+3}, x_{n+3}) + ... + G(x_{m-1}, x_m, x_m) \\ &\preceq [k^n + k^{n+1} + k^{n+2} + k^{n+3} + \cdots + k^{m-1}]G(x_0, x_1, x_1) \\ &\preceq [\frac{k^n}{e - k}]G(x_0, x_1, x_1). \end{aligned}$$

Using Lemmas 1.4 and 1.5, we have

$$\begin{aligned} r\left(\frac{k^n}{e - k}\right) &\leq r(k^n).r((e - k)^{-1}) \\ &\leq \frac{(r(k))^n}{e - r(k)} \end{aligned}$$

By Remark 1.2 and Lemma 1.7,

$$\|\frac{k^n}{e - k}\| \to 0 \text{ as } n \to \infty.$$

It follows that for any $c \in A$ with $\theta \ll c$, there exist $N \in \mathbb{N}$ such that $m > n > N$, we have that

$$G(x_n, x_m, x_m) \preceq k^n(e - k)^{-1}G(x_0, x_1, x_1) \ll c$$

which implies that $\{x_n\}$ is Cauchy. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$ and $Tx^* \in X$
By continuity, we have

$$\begin{aligned}\lim_{n\to\infty} G(x_{n+1}, x_n, Tx^*) &= G(\lim_{n\to\infty} x_{n+1}, \lim_{n\to\infty} x_n, Tx^*)\\ &= G(x^*, x^*, Tx^*)\end{aligned}$$

Thus $Tx^* = x^*$.
To prove the uniqueness of x^*, suppose y^* is another fixed point of T such that $x^* \neq y^*$. Then

$$\begin{aligned}G(x^*, x^*, y^*) &= G(Tx^*, Tx^*, Ty^*)\\ &\preceq kG(x^*, x^*, y^*) \text{ (a contradiction)}\end{aligned}$$

So, $x^* = y^*$. Hence, the fixed point is unique.

Corollary 2.8
Let (X, G) be a complete G-cone metric space over a Banach space A and P be a non-normal cone. Suppose that the mapping $T : X \to X$ satisfies the generalized contraction condition

$$G(Tx, Ty, Tz) \preceq kG(x, y, z), \forall x, y \in X$$

where $0 <\| k \|< 1, k \in P, x, y, z \in X$. Then T has a unique fixed point in X.

Proof:
The proof holds from the proof of Theorem 2.7 above as $r(k) \leq\| k \|$.

Corollary 2.9
Let (X, G) be a complete G- metric space. Suppose that the mapping $T : X \to X$ satisfies the generalized contraction condition

$$G(Tx, Ty, Tz) \leq kG(x, y, z), \forall x, y \in X$$

where $0 < k < 1, x, y, z \in X$. Then T has a unique fixed point in X.

Proof:
The proof holds from the proof of Theorem 2.7 above for $A = \mathbb{R}$ and $r(k) = k$

Remark 2.10

(i) In Theorem 2.7, we only suppose that the spectral radius of k is less than 1, neither $k \prec e$ nor $\|k\| < 1$ is assumed. This is vital. In fact, the condition $r(k) < 1$ is weaker than that $\|k\| < 1$, as is illustrated by (Hao and Shaoyuan [18] Example 2.1). Thus if Theorem 2.7 holds then Corollary 2.8 and Corollary 2.9 holds. The reverse is untrue.

(ii) If $A = \mathbb{R}$, $y = z$ and $G(x, y, y) = d(x, y)$, then Theorem 2.7 becomes the Banach contraction principle in Banach [23].

Theorem 2.11
Let (X, G) be a complete and bounded G-cone metric space over a Banach algebra A and P a cone. Suppose that the mapping $T : X \to X$ satisfies the generalized Lipschitz condition

$$G(Tx, Ty, z) \preceq aG(x, y, z) + bG(x, Tx, z) + cG(y, Ty, z) + dG(Tx, Ty, z)$$

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for all $x, y, z \in X$, where $0 < r(a) + r(b) + r(c) + r(d) < 1, a, b, c, d \in P, x, y, z \in X$. Then T has a unique fixed point in X.

Proof:
Choose $x_0 \in X$ and set $x_n = T^n x_0$, $n \geq 1$. We have

$$\begin{aligned} G(x_{n+1}, x_n, z) &= G(Tx_n, Tx_{n-1}, z) \\ &\preceq aG(x_n, x_{n-1}, z) + bG(x_n, Tx_n, z) + cG(x_{n-1}, Tx_{n-1}, z) + dG(Tx_n, Tx_{n-1}, z) \end{aligned}$$

Clearly,

$$G(x_{n+1}, x_n, z) \preceq [\frac{a+c}{e-b-d}]G(x_n, x_{n-1}, z)$$

Hence

$$G(x_{n+1}, x_n, z) \preceq [\frac{a+c}{e-b-d}]^n G(x_1, x_0, z)$$

Set $q = \frac{a+c}{e-b-d}$, so that

$$G(x_{n+1}, x_n, z) \preceq q^n G(x_1, x_0, z)$$

Let K be G bound for X. For $x_p \in X, 0 \leq p \leq n$, we have

$$\| G(x_1, x_0, x_p) \| \preceq K$$

By the rectangle inequality with $n > m$, we obtain

$$\begin{aligned} G(x_n, x_m, z) &\preceq G(x_n, x_{n-1}, x_{n-1}) + G(x_{n-1}, x_{n-2}, x_{n-2}) + \cdots + G(x_{m+1}, x_m, z) \\ &\preceq q^{n-1}K + q^{n-2}K + q^{n-3}K + \cdots + q^{m+2}K + q^{m+1}G(x_1, x_0, z) \\ &\preceq (q^{n-1} + q^{n-2} + q^{n-3} + \cdots + q^{m+2})K + q^{m+1}G(x_1, x_0, z) \\ &\preceq (q^{m+2} + q^{m+3} + \cdots + q^{n-1})K + q^{m+1}G(x_1, x_0, z) \\ &\preceq (1 + q + q^2 + \cdots + q^{n-m-3})q^{m+2}K + q^{m+1}G(x_1, x_0, z) \\ &\preceq Kq^{m+2} \sum_{i=0}^{n-m-3} q^i + q^{m+1}G(x_1, x_0, z) \\ &\preceq Kq^{m+2}(e-q)^{-1} + q^{m+1}G(x_1, x_0, z) \end{aligned}$$

Using Lemmas 1.4 and 1.5, we have

$$\begin{aligned} r(q^{m+2}(e-q)^{-1})K &\leq r((e-q)^{-1}).(r(q))^{m+2} \\ &\leq \frac{(r(q))^{m+2}}{1-r(q)} \\ &< 1 \\ &\text{and} \\ r(q^{m+1}) &\leq (r(q))^{m+1} \end{aligned}$$

By Remark 1.2 and Lemma 1.7,

$$\|q^{m+2}(e-q)^{-1}\| \rightarrow 0, \|q^{m+1}\| \rightarrow 0 \text{ as } m \rightarrow \infty$$

It follows that for any $c \in A$ with $\theta \ll c$, there exist $N \in \mathbb{N}$ such that $m > n > N$, we have that

$$G(x_n, x_m, x_m) \preceq k^n(e-k)^{-1}G(x_0, x_1, x_1) \ll c$$

which implies that $\{x_n\}$ is Cauchy. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$ and $Tx^* \in X$
Then we get

$$\begin{aligned} \|\frac{q^{m+2}}{e-q}K\| &\to 0, \|q^{m+1}G(x_1, x_0, z)\| \\ &\to 0 \text{ as } m \to \infty. \end{aligned}$$

By Lemma 1.6 and for any $c \in A$ with $\frac{c}{2} \gg \theta$, $n > m > N^1, N^1 \in N$, we have

$$\begin{aligned} G(x_n, x_m, z) &\preceq [\frac{q^{m+2}}{e-q}]K + q^{m+1}G(x_0, x_1, z) \\ &\ll \frac{c}{2} + \frac{c}{2} = c \end{aligned}$$

which implies that $\{x_n\}$ is Cauchy. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$ and $Tx^* \in X$
By continuity, we have

$$\begin{aligned} \lim_{n\to\infty} G(x_{n+1}, x_n, Tx^*) &= G(\lim_{n\to\infty} x_{n+1}, \lim_{n\to\infty} x_n, Tx^*) \\ &= G(x^*, x^*, Tx^*) \end{aligned}$$

Thus $Tx^* = x^*$.
For uniqueness, suppose y^* is another fixed point of T such that $x^* \neq y^*$. Then

$$\begin{aligned} G(x^*, x^*, y^*) &= G(Tx^*, Tx^*, Ty^*) \\ G(Tx^*, Tx^*, Ty^*) &\preceq aG(x^*, x^*, y^*) + bG(x^*, Tx^*, y^*) + dG(Tx^*, Ty^*, y^*) \\ G(x^*, x^*, y^*) &\leq aG(x^*, x^*, y^*) + bG(x^*, x^*, y^*) + dG(x^*, y^*, y^*) \\ (e-a-b-d)G(x^*, x^*, y^*) &\preceq \theta \text{ (a contradiction)} \end{aligned}$$

So, $x^* = y^*$. Hence, the fixed point is unique.

Corollary 2.12
Let (X, G) be a complete and bounded G-cone metric space over a Banach space A and P a cone. Suppose that the mapping $T : X \to X$ satisfies the generalized Lipschitz condition

$$G(Tx, Ty, z) \preceq aG(x, y, z) + bG(x, Tx, z) + cG(y, Ty, z) + dG(Tx, Ty, z)$$

for all $x, y, z \in X$, where $0 <\| a \| + \| b \| + \| c \| + \| d \|< 1, a, b, c, d \in P, x, y, z \in X$. Then T has a unique fixed point in X.

Proof
The proof follows from Theorem 2.11 above as $\| a \| + \| b \| + \| c \| + \| d \| \geq r(a)+r(b)+r(c)+r(d)$

Corollary 2.13
Let (X, G) be a complete G-cone metric space. Suppose that the mapping $T : X \to X$ satisfies the generalized Lipschitz condition

$$G(Tx, Ty, z) \leq aG(x, y, z) + bG(x, Tx, z) + cG(y, Ty, z) + dG(Tx, Ty, z)$$

for all $x, y, z \in X$, where $0 < a + b + c + d < 1, a, b, c, d \in \mathbb{R}, x, y, z \in X$. Then T has a unique fixed point in X. and for any $x \in X$.

Proof
The proof follows from Theorem 2.11 above with $A = \mathbb{R}$ and $r(a) = a, r(b) = b, r(c) = c$ and $r(d) = d$

Remark 2.14
Theorems 2.7 and 2.11 above are significant extension of fixed point results of the contraction type maps on G-metric spaces. It is significant because it cannot be obtained by a non-linear scalarization of the map defined on the G-metric spaces. This is unlike the map defined on G-cone metric space over Banach space which can be obtained from a non-linear scalarization of the map defined on the G-metric spaces thus rendering the fixed point results of such map a mere extension of the map on the G-metric space.

3 Application

It is not an uncommon practice to generalize/extend existing results. But it is essential that new results are useful for solving/analyzing problems that were hitherto unsolved/not analyzed by the results they generalize or extend. The idea in such research effort is to get results that probably accommodate more problems. Thus, an application of the result of Theorem 2.7 is hereby presented.
Theorem 3.1
Consider the first-order periodic boundary problem given as:

$$y(t) = \epsilon + \int_0^t F(\tau, y(\tau))d\tau$$

where F is a continuous function on $[-r, r] \times [\epsilon - \delta, \epsilon + \delta]$. Suppose $y_1, y_2 \in [\epsilon - \delta, \epsilon + \delta]$ if $\mid x \mid \leq r$, induces $\mid F(x, y_1) - F(x, y_2) \mid \preceq L \mid y_1 - y_2 \mid$ for the continuous function $F(x, y)$. Then there exists a unique solution of Equation 3 if

$$r \quad < \quad \min\{\frac{\delta}{2(\max_{[-r,r]\times[\epsilon-\delta,\epsilon+\delta]} \mid F(x,y) \mid}, \frac{1}{L}\}$$

Proof:
Let $Y = A = C([-r, r])$ and $P = \{u \in A : v \geq 0\}$. Suppose $G : Y^3 \to A$ such that $G(y_1, y_2, y_3)(t) = \max_{-r \leq t \leq r}(\sum_{i=1}^{3} \sum_{i<j} \mid y_i - y_j \mid e^t)$ with $y_i : [-r, r] \to \mathbb{R}$ for each $i = 1, 2, 3$. It is clear that (Y, G) is a complete G-cone metric space over Banach algebra. We now prove the existence of solution to Equation 3 by proving the existence of fixed of the map $T : C([-r, f]) \to \mathbb{R}$ defined by

$$Ty(t) = \epsilon + \int_0^t F(\tau, y(\tau))d\tau$$

Define for any $g : [-r, r] \to \mathbb{R}$

$$B(\epsilon, \delta g) \triangleq \{\psi(t) \in C([-r, r]) : G(\epsilon, \epsilon, \psi) \leq \delta g\}.$$

If $x(t), y(t) \in B(\epsilon, \delta g)$, then

$$\begin{aligned}
G(Tx,Tx,Ty)(t) &= 2 \mid Tx - Ty \mid e^t \\
&= 2(\max_{-r\leq\tau\leq r} \mid \int_0^t F(\tau, x(\tau))d\tau - \int_0^t F(\tau, y(\tau))d\tau \mid) e^t \\
&\preceq 2r \max_{-r\leq\tau\leq r} \mid F(\tau, x(\tau)) - F(\tau, y(\tau)) \mid e^t \\
&\preceq 2rL \max_{-r\leq\tau\leq r} \mid x(\tau) - y(\tau)) \mid e^t \\
&= rLG(x,x,y)(t), \\
\text{and} \\
G(Tx,Tx,\epsilon)(t) &= 2 \max_{-r\leq\tau\leq r} \mid \int_0^t F(\tau, x(\tau))d\tau \mid e^t \\
&\preceq 2r \max_{-r\leq\tau\leq r} \mid F(\tau, x(\tau)) \mid e^t \\
&\preceq 2rMe^t \text{ where } M = \max_{-r\leq\tau\leq r} \mid F(\tau, x(\tau)) \mid) \\
&\preceq \delta g(t),
\end{aligned}$$

Since (Y, G) is a complete space, there is $x^* \in Y$ such that $x_n \to x^*$ as $(n \to \infty)$. So, for each $c \in intP$, there exists N; whenever $n > N$, we obtain $G(x_n, x_n, x) \ll \frac{c}{2}$. Thus, it follows from

$$\begin{aligned}
G(x,x,\epsilon) &\preceq 2G(x,x,x_n) + G(x_n,x_n,\epsilon) \\
&\preceq \delta g + c
\end{aligned}$$

and by the property: $a \leq b$ if $a \leq b + c$ for each $c \int P$ and $a, b \in P$, that $G(x,x,\epsilon) \preceq \delta g$, , which means $x \in B(\epsilon, \delta g)$ i.e $x \in (B(\epsilon, \delta g), G)$ is complete.

Conclusion In this present paper, we introduced the G-cone metric space over Banach algebra and generalized contraction map on G-cone metric space over Banach algebra. The fixed point results of the space is shown to generalize the corresponding fixed point results of metric space, G-metric space and G-cone metric space. Also the results of Theorem 2.7 and Theorem 2.11 are not obtainable from either Corollary 2.8 or Corollary 2.9 and Corollary 2.12 or Corollary 2.13 respectively or the results they generalize.

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Competing financial interests

The authors declare no competing financial interests.

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