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Asset Optimization Problem In A Financial Institution

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Abstract

This paper looked at how a financial institution could optimally allocate its total wealth among three assets namely; treasury, security and loan in a stochastic interest rate setting. The optimal investment strategy was derived through the application of a stochastic optimization theory for the case of constant relative risk aversion (CRRA) utility function. Next,numerical examples using published data obtained from CBN statistical bulletin and Nigeria Stock Exchange Fact Book was presented to illustrate the dynamics of the optimal investment strategy. From the results it was seen that the optimal investment strategy was to shift the financial institution investment away from the risky assets (security and loan) toward the riskless asset (treasury). Also the investment in security and loan was observed to be more risky as the volatility increased.The results further showed that there is increased investment in the risky assets as the investor became less risk averse.

Keywords: optimal investment, investment strategy, stochastic optimization theory, stochastic interest rate, portfolio.
MSC2010: 93E20, 60H10, 62P05, 91B70, 49K45

1 Introduction

The need for financial institutions to optimally invest in assets or optimally allocate its total wealth among assets is very important. For instance, if the return on a particular loan turns out to be very high at the end of the loan contract period, the financial institution might regret not having allocated a large enough portion of its capital to such loan. Therefore, a dynamic portfolio or optimal asset allocation is very crucial in a financial institution management. In recent time, interest in this topic in stochastic framework has grown commensurately ([1], [2], [3]). An important issue in assets allocation problem is how to characterize the optimal rebalancing pattern of assets through time ([2], [4]). Maximization of the expected utility from terminal wealth has been the method of

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dealing with the above problem.
Grant and Peter (2014) studied an optimal assets allocation problem with stochastic interest rates which takes into account specific features of bank [3]. Their work presents a numerical aspect of the derived Hamilton – Jacobi – Bellman (HJB) equation and also looked at the optimal assets allocation problem from a practical viewpoint. Fouche *et al.* (2006) also considered assets allocation problem [5]. They illustrated that it is possible to use an analytic approach to optimize assets allocation strategies for banks. They formulated an optimal bank valuation problem through optimal choices of loan rate and demand which leads to maximal deposits, provisions for deposits withdrawals and bank profitability subject to cash flow, loan demand, financing and balance sheet constraints.
Several studies have also investigated the optimal assets allocation problems using stochastic control theory developed by Merton ([6], [7]) in discrete and continuous time setting ([8], [9], [10]). The approach solved nonlinear partial differential Hamilton – Jacobi – Bellman equation to find the closed form solution for the value function. Astic and Tourin (2014) considered a bank that invests in both liquid and illiquid assets [11]. The goal of the investor is to maximize its shareholders' profit while satisfying some regulatory constraints. They studied the sensitivity of the shareholders' gain and optimal portfolio allocations, and the associated bondholders' payoff to the minimal capital requirement and liquidity ratio. In their research, they found that tightening the liquidity constraint adversely affects the rates of return on investment while preventing some large losses that occur when the portfolio is very illiquid and stiffening the minimal capital requirement penalizes the shareholders but seems to have little influence on the bondholders.
The motivation for the current study lies in the work of Grant and Peter (2014), who looked at how a financial institution can optimally allocate its wealth among its assets namely; treasury, security and loan, and also manage its capital under stochastic interest rates. The current study modified the existing security and loan models, formulates assets optimization problem, estimate the parameters of the models using data obtained from CBN statistical bulletin (1980 -2010)and Nigeria Stock Exchange Fact Book (2005 – 2010) by maximum likelihood method ([14], [15]) and applied the models to a financial institution in Nigeria.

2 Formulation of the optimization problem and its transformation into partial differential equation

2.1 The assets model in the financial market for the financial institution

We assume that the financial institution can invest its wealth in a market consisting of three assets. The first asset in the financial market is a riskless treasury and its price at time t can be denoted by $S_0(t)$. It evolves according to the following stochastic differential equation:

$$\frac{dS_0(t)}{S_0(t)} = r(x)dt, S_0(0) = s_0. \tag{2.1}$$

The dynamics of the short rate process $r(t)$, is given by the stochastic differential equation described by Affine model:

$$dr(t) = (a - br(t))dt + \sqrt{k_1 r(t) + k_2}\,dw_r(t), r(0) = r_0. \tag{2.2}$$

Let $\sigma_r = \sqrt{k_1 r(t) + k_2}$, then from (2) we have

$$dr(t) = (a - br(t))dt + \sigma_r dw_r(t), r(0) = r_0. \tag{2.3}$$

where a, b, k_1 and k_2 are constants. The second asset in the financial market is a risky security whose price is denoted by $S(t)$, $t \geq 0$. Its dynamics can be described by the equation:

$$\frac{dS(t)}{S(t)} = (r(t) + \nu\sigma_1 + \sigma_p\lambda_r k_1 r(t))dt + \sigma_p\sigma_r\sqrt{r(t)}\,dw_r(t) + \sigma_1 dw_s(t). \tag{2.4}$$

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From equation (2.4), if we assume that the volatility scale factor σ_p which measures how the risk sources of interest rate affect the price of the security is equal to zero (the risk sources of the interest rate have no effect on the price of the security) then the modified security model is given by:

$$\frac{dS(t)}{S(t)} = (r(t) + \nu\sigma_1 + 0)dt + 0 + \sigma_1 dw_s(t) \tag{2.5}$$

Therefore, equation (2.5) becomes

$$\frac{dS(t)}{S(t)} = (r(t) + \nu\sigma_1)dt + \sigma_1 dw_s(t), S(0) = s_0 \tag{2.6}$$

where ν and σ_1 are constants. Let $\nu\sigma_1 = \lambda_s$, and $\sigma_1 = \sigma_s$ then (2.6) becomes

$$\frac{dS(t)}{S(t)} = (r(t) + \lambda_s)dt + \sigma_s dw_s(t), S(0) = s_0 \tag{2.7}$$

The third asset is a loan to be amortized over a period $[0, T]$ whose price at time $t \geq 0$ is denoted by $L(t)$. Let us assume that the price of the asset can be described by a stochastic differential equation similar to (2.7) above. Then

$$\frac{dL(t)}{L(t)} = (r(t) + \lambda_l)dt + \sigma_l dw_l(t), L(0) = l_0 \tag{2.8}$$

where λ_l and σ_l are constants.

Here we assume that there is no correlation between $w_s(t)$ and $w_r(t)$, between $w_l(t)$ and $w_r(t)$ and between $w_s(t)$ and $w_l(t)$.

2.2 The derivation of the financial institution assets portfolio model

Let $X(t)$ denote the value of the financial institution assets portfolio at time $t \in [0, T]$, $\pi_s(t)$ and $\pi_l(t)$ denote the amount invested in the security and loan respectively. Therefore, $\pi_0(t) = X(t) - \pi_s(t) - \pi_l(t)$ denotes the amount invested in the riskless asset. The assets portfolio model is given by the following SDE:

$$\begin{aligned} dX(t) &= (X(t) - \pi_s(t) - \pi_l(t))\frac{dS_0(t)}{S_0(t)} + \pi_s(t)\frac{dS(t)}{S(t)} + \pi_l(t)\frac{dL(t)}{L(t)} \\ &= (X(t)r(t) + \pi_s(t)\lambda_s + \pi_l(t)\lambda_l)dt + \pi_s(t)\sigma_s dw_s(t) \\ &\quad + \pi_l(t)\sigma_l dw_l(t). \end{aligned}$$

Definition (Admissible strategy)
An investment strategy $\pi(t) = (\pi_s(t), \pi_l(t))$ is said to be admissible if the following conditions are satisfied.

(i) $\pi_s(t)$ and $\pi_l(t)$ are all f_t-measurable.

(ii) $E\left(\int_0^T (\pi_s^2(t)\sigma_s^2 + \pi_l^2(t)\sigma_l^2)dt\right) < \infty$

(iii) The stochastic differential equation (2.9) has a unique solution $\forall \pi(t) = (\pi_s(t), \pi_l(t))$.

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2.3 The formulation of the financial institution's asset portfolio optimization problem

Let the set of all admissible strategies be denoted by Π. Under the asset portfolio (2.9), the financial institution looks for an optimal investment strategy $\pi_s^*(t)$ and $\pi_l^*(t)$ which maximizes the expected utility of the terminal wealth. i.e.

$$\max_{\pi(t)\in\Pi} E[U(X(T))] \tag{2.9}$$

Based on the classical tools of stochastic optimal control, we state the optimization problem as follows:
Maximize: $E[U(X(T))]$
Subject to the following constraints
$dr(t) = (a - br(t))dt + \sigma_r dw_r(t)$
$dX(t) = (X(t)r(t) + \pi_s(t)\lambda_s + \pi_l(t)\lambda_l)dt + \pi_s(t)\sigma_s dw_s(t) + \pi_l(t)\sigma_l dw_l(t)$, $0 \le t \le T$ where $X(0) = x_0$ and $r(0) = r_0$ are the initial conditions of the optimization problem.

The objective is to maximize the expected utility of the financial institution's portfolio at future date $T > 0$. That is, find the optimal value function

$$H(t, r, x) = \max_{\pi(t)\in\Pi} E[U(X(T))|r(t) = r, X(t) = x] \tag{2.10}$$

and the optimal strategy $\pi^*(t) = (\pi_s^*(t),\, \pi_l^*(t))$ such that

$$H_{\pi^*(t)}(t, r, x) = H(t, r, x). \tag{2.11}$$

2.4 The transformation of the optimization problem into partial differential equation

The Hamilton – Jacobi – Bellman equation associated with the asset portfolio optimization problem is:

$$\begin{aligned}\max_{\pi(t)\in\Pi}\{H_t \quad &+ \quad [X(t)r(t) + \pi_s(t)\lambda_s + \pi_l(t)\lambda_l]H_x + \frac{1}{2}(\pi_s^2(t)\sigma_s^2 + \pi_l^2(t)\sigma_l^2)H_{xx}\\ &+(\pi_s(t)\sigma_s\sigma_r + \pi_l(t)\sigma_l\sigma_r)H_{xr} + [a - br(t)]H_r +\\ &\frac{1}{2}\sigma_r^2 H_{rr}\} = 0\end{aligned}$$

$$H(T, r, x) = U(x) \tag{2.12}$$

where H_t, H_x, H_r, H_{xx}, H_{rr} and H_{xr} denote partial derivatives of first and second orders with respect to t, r and x respectively.

The first order maximizing conditions for the optimal investment strategy $(\pi_s^*(t), \pi_l^*(t))$ (i.e., differentiating (2.13) with respect to $\pi_s(t)$ and $\pi_l(t)$) gives

$$\lambda_s H_x + \pi_s(t)\sigma_s^2 H_{xx} + \sigma_s\sigma_r H_{xr} = 0 \tag{2.13}$$

$$\lambda_l H_x + \pi_s(t)\sigma_l^2 H_{xx} + \sigma_l\sigma_r H_{xr} = 0 \tag{2.14}$$

respectively. Next, we solve (2.15) and (2.16) for $\pi_s(t)$ and $\pi_l(t)$ to obtain the optimal strategy $(\pi_s^*(t), \pi_l^*(t))$.

From equations (2.15) and (2.16) we have

$$\pi_s^*(t) = -\frac{\lambda_s H_x)}{\sigma_s^2 H_{xx}} - \frac{\sigma_r H_{xr})}{\sigma_s H_{xx}}, \pi_l^*(t) = -\frac{\lambda_l H_x}{\sigma_l^2 H_{xx}} - \frac{\lambda_r H_{xr}}{\sigma_l H_{xx}}. \tag{2.15}$$

Substituting (2.17) into (2.13) gives the partial differential equation (PDE) for the value function $H(t,r,x)$.

$$\begin{aligned} H_t + xrH_x \quad &- \quad \left(\frac{\lambda_s^2}{2\sigma_s^2} + \frac{\lambda_l^2}{2\sigma_l^2}\right)\frac{H_x^2}{H_{xx}} - \frac{\sigma_r^2 H_{xr}^2}{H_{xx}} - \left(\frac{\lambda_s\sigma_r}{\sigma_s} + \frac{\lambda_l\sigma_r}{\sigma_l}\right)\frac{H_x H_{xr}}{H_{xx}} \\ &+(a-br)H_r + \frac{1}{2}\sigma_r^2 H_{rr} = 0 \end{aligned}$$

After simplification, the HJB equation (2.13) is equivalent to the partial differential equation (2.18). The problem now is to solve (2.18) for the value function $H(t,r,x)$ and replace it in (2.17).

3 The assets portfolio optimization problem and its solution under power utility function

From (2.17) and considering constant relative risk aversion (CRRA) utility function: $U(x) = \frac{x^\beta}{\beta}$, $\beta < 1$, $\beta \neq 0$, $\beta =$Risk aversion parameter
show that the value function H has the following form:

$$H(t,r,x) = \frac{x^\beta}{\beta} f(t,r), \beta < 1, \beta \neq 0 \tag{3.1}$$

With the boundary condition:

$$f(T,r) = 1 \forall r. \tag{3.2}$$

From (3.1)

$$\begin{pmatrix} H_t = \frac{x^\beta}{\beta} f_t, H_x = x^{\beta-1} f, H_{xx} = (\beta-1)x^{\beta-2} f \\ H_r = \frac{x^\beta}{\beta} f_r, H_{rr} = \frac{x^\beta}{\beta} f_{rr}, H_{xr} = x^(\beta-1) f_r. \end{pmatrix}$$

Where H_t, H_x, H_r, H_{xx}, H_{xr} and H_{rr} are first order and second order partial derivatives of H with respect to t and r. f_t, f_r and f_{rr} represent the first order and second order partial derivatives of f with respect to t and r.
Therefore, introducing the derivatives in (3.3) into (2.18) gives

$$\begin{aligned} \frac{x^\beta}{\beta} f_t \quad &+ \quad rx^\beta f - \left(\frac{\lambda_s^2}{2\sigma_s^2} + \frac{\lambda_l^2}{2\sigma_l^2}\right)\frac{x^\beta f}{(\beta-1)}) - \sigma_r^2 \frac{x^\beta f_r^2}{(\beta-1)f} - \left(\frac{\lambda_s\sigma_r}{\sigma_s} + \frac{\lambda_l\sigma_r}{\sigma_l}\right)\frac{x^\beta f_r}{(\beta-1)} \\ &+(a-br)\frac{x^\beta}{\beta} f_r + \frac{1}{2}\sigma_r^2 \frac{x^\beta}{\beta} f_{rr} = 0 \end{aligned}$$

So that,

$$\begin{aligned} f_t \quad &+ \quad \left[r\beta - \left(\frac{\beta\lambda_s^2}{2\sigma_s^2(\beta-1)} + \frac{\beta\lambda_l^2}{2\sigma_l^2(\beta-1)}\right)\right] f - \frac{\beta\sigma_r^2 f_r^2}{(\beta-1)f} \\ &+ \left[(a-br) - \left(\frac{\beta\lambda_s\sigma_r}{\sigma_s(\beta-1)} + \frac{\beta\lambda_l\sigma_r}{\sigma_l(\beta-1)}\right)\right] f_r + \frac{1}{2}\sigma_r^2 f_{rr} = 0 \end{aligned}$$

Next we conjecture $f(t,r)$ as the following:

$$f(t,r) = A(t)exp(\phi(t)r), A(T) = 1, \Phi(T) = 0 \tag{3.3}$$

From (3.6)

$$\begin{pmatrix} f_t = (A_1'(t) + r\phi'(t)A(t))exp(\phi(t)r) \\ f_r = \phi(t)A(t)exp(\phi(t)r), f_{rr} = \phi^2(t)A(t)exp(\phi(t)r). \end{pmatrix}$$

Hence substituting for f_t, f_r and f_{rr} in (3.7) and noting that $f = A(t)exp(\phi(t)r)$ gives

$$\begin{aligned} rA(t)exp(\phi(t)r)(\phi'(t) \quad &+ \quad \beta - b\phi(t)) + exp(\phi(t)r)[A_1'(t) \\ &+ \left(\frac{1}{2}\sigma_r^2 - \frac{\beta\sigma_r^2}{\beta-1}\right)\phi^2(t)A(t) \\ &+ \left[a - \left(\frac{\beta\lambda_s\sigma_r}{\sigma_s(\beta-1)} + \frac{\beta\lambda_l\sigma_r}{\sigma_l(\beta-1)}\right)\right]\phi(t)A(t) - \\ &\left(\frac{\beta\lambda_s^2}{2\sigma_s^2(\beta-1)} + \frac{\beta\lambda_l^2}{2\sigma_l^2(\beta-1)}\right)A(t)] = 0 \end{aligned}$$

By classical variable decomposition approach, we decompose (3.8) into

$$\phi'(t) + \beta - b\phi(t) = 0 \tag{3.4}$$

$$\begin{aligned} A_1'(t) \quad &+ \quad \left(\frac{1}{2}\sigma_r^2 - \frac{\beta\sigma_r^2}{(\beta-1)}\right)\phi^2(t)A(t) + \left[a - \left(\frac{\beta\ lambda_s\sigma_r}{\sigma_s(\beta-1)} + \frac{\beta\lambda_l\sigma_r}{\sigma_l(\beta-1)}\right)\right]\phi(t)A(t) \\ &- \left(\frac{\beta\lambda_s^2}{2\sigma_s^2(\beta-1)} + \frac{\beta\lambda_l^2}{2\sigma_l^2(\beta-1)}\right)A(t) = 0 \end{aligned}$$

Now, we solve for $\phi(t)$ in equation (3.9). From (3.9) we have

$$\phi'(t) - b\phi(t) = -\beta. \tag{3.5}$$

Hence

$$\phi(t) = \frac{\beta}{b} + ce^{bt}. \tag{3.6}$$

From equation (3.6), $\phi(T) = 0$. Therefore, from equation (3.12) we have

$$c = -\frac{\beta e^{-bT}}{b}$$

Hence, from equation (30), we have

$$\phi(t) = \frac{\beta}{b}(1 - e^{-b(T-t)}). \tag{3.7}$$

Next we solve for $A(t)$ in (3.10). From (3.1), $\beta < 1$. Hence, from (3.10) we have

$$\begin{aligned} A_1'(t) \quad &+ \quad \left(\frac{1}{2}\sigma_r^2 + \frac{\beta\sigma_r^2}{(1-\beta)}\right)\phi^2(t)A(t) \\ &+ \left[a + \left(\frac{\beta\lambda_s\sigma_r}{\sigma_s(1-\beta)} + \frac{\beta\lambda_l\sigma_r}{\sigma_l(1-\beta)}\right)\right]\phi(t)A(t) \\ &+ \left(\frac{\beta\lambda_s^2}{2\sigma_s^2(1-\beta)} + \frac{\beta\lambda_l^2}{2\sigma_l^2(1-\beta)}\right)A(t) = 0 \end{aligned}$$

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Let

$$\begin{aligned} p(t) &= \left(\frac{1}{2}\sigma_r^2 + \frac{\beta\sigma_r^2}{(1-\beta)}\right)\phi^2(t) + \left[a + \left(\frac{\beta\lambda_s\sigma_r}{\sigma_s(1-\beta)} + \frac{\beta\lambda_l\sigma_r}{\sigma_l(1-\beta)}\right)\right]\phi(t) \\ &+ \left(\frac{\beta\lambda_s^2}{2\sigma_s^2(1-\beta)} + \frac{\beta\lambda_l^2}{2\sigma_l^2(1-\beta)}\right) \end{aligned}$$

then from equation (3.14), we have the following

$$\frac{dA(t)}{dt} + p(t)A(t) = 0. \tag{3.8}$$

Solving equation (3.15) and imposing the boundary condition $A(T) = 1$ gives

$$A(t) = exp(P(T) - P(t)). \tag{3.9}$$

Hence,

$$\begin{aligned} f(t,r) &= A(t)exp(\phi(t)r) \\ &= exp\left((P(T) - P(t)) + \frac{\beta}{b}(1 - e^{-b(T-t)})r\right) \end{aligned}$$

and

$$H(t,r,x) = \frac{x^\beta}{\beta}exp\left((P(T) - P(t)) + \frac{\beta}{b}(1 - e^{-b(T-t)})r\right) \tag{3.10}$$

Theorem 1
Given (2.17), (3.3) and (3.7), the optimal investment strategy under stochastic interest rate framework and CRRA utility function is given by:

$$\pi_s^*(t) = \left(\frac{\lambda_s}{\sigma_s^2(1-\beta)}\right)X(t) + \left(\frac{\beta\beta_1\sigma_r}{\sigma_s b(1-\beta)}\right)X(t)$$

$$\pi_l^*(t) = \left(\frac{\lambda_l}{\sigma_l^2(1-\beta)}\right)X(t) + \left(\frac{\beta\beta_1\sigma_r}{\sigma_l b(1-\beta)}\right)X(t)$$

$$\begin{aligned} \pi_0^*(t) &= X(t) - X(t)\left[\left(\frac{\lambda_s}{\sigma_s^2(1-\beta)}\right) + \left(\frac{\beta\beta_1\sigma_r}{\sigma_s b(1-\beta)}\right)\right] \\ &-X(t)\left[\left(\frac{\lambda_l}{\sigma_l^2(1-\beta)}\right) + \left(\frac{\beta\beta_1\sigma_r}{\sigma_l b(1-\beta)}\right)\right] \end{aligned}$$

Therefore, the optimal proportion of wealth invested in security, loan and treasury are:

$$\pi_{sp}^*(t) = \left(\frac{\lambda_s}{\sigma_s^2(1-\beta)}\right) + \left(\frac{\beta\beta_1\sigma_r}{\sigma_s b(1-\beta)}\right)$$

$$\pi_{lp}^*(t) = \left(\frac{\lambda_l}{\sigma_l^2(1-\beta)}\right) + \left(\frac{\beta\beta_1\sigma_r}{\sigma_l b(1-\beta)}\right)$$

$$\begin{aligned} \pi_{0p}^*(t) &= 1 - \left[\left(\frac{\lambda_s}{\sigma_s^2(1-\beta)}\right) + \left(\frac{\beta\beta_1\sigma_r}{\sigma_s b(1-\beta)}\right)\right] \\ &- \left[\left(\frac{\lambda_l}{\sigma_l^2(1-\beta)}\right) + \left(\frac{\beta\beta_1\sigma_r}{\sigma_l b(1-\beta)}\right)\right] \end{aligned}$$

where $\beta_1 = (1 - e^{-b(T-t)})$.

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4 Numerical examples

Here, we present the numerical simulation for the evolution of the optimal investment strategy derived in the previous section. We take the investment period $T = 10$ years, $\beta = 0.5$ and assumed that $\lambda_l = 0.0031$, $\sigma_l = 0.0874$. The remaining parameters $b = 2.5148$, $\lambda_s = 0.0022$, $\sigma_s = 0.0748$, $\sigma_r = 0.3535$ are estimated from data obtained from CBN statistical bulletin and Nigeria Stock Exchange Fact Book.

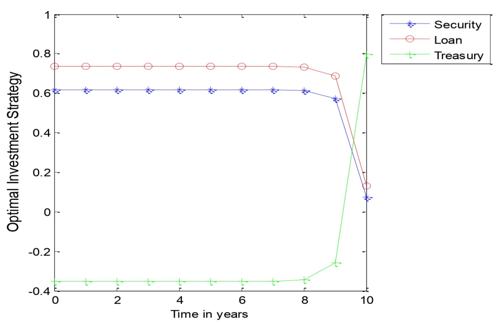

Figure 1:: The effect of time on the optimal investment strategy

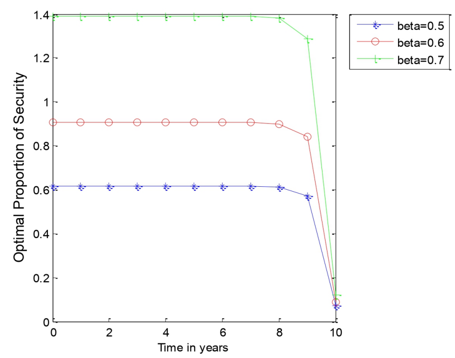

Figure 2:: The effect of β on the optimal investment strategy $\pi_s^*(t)$ for security

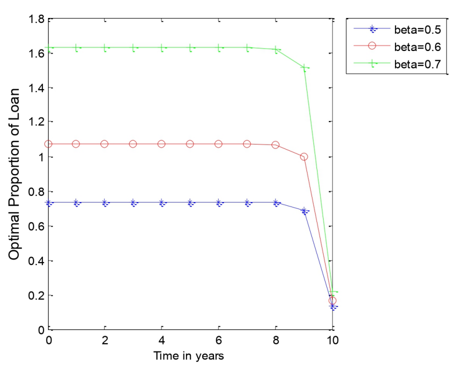

Figure 3:: The effect of β on the optimal investment strategy $\pi_l^*(t)$ for loan

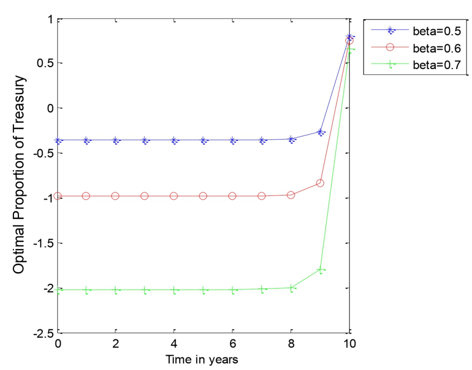

Figure 4: The effect of β on the optimal investment strategy $\pi_0^*(t)$ for treasury

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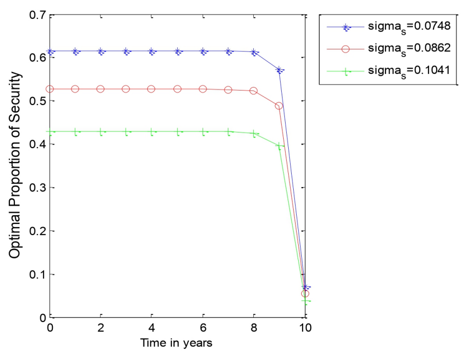

Figure 5: The effect of σ_s on the optimal investment strategy $\pi_s^*(t)$ for security

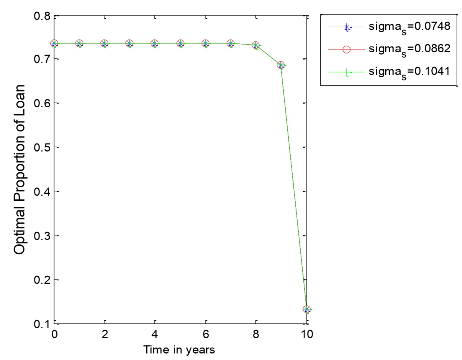

Figure 6: The effect of σ_s on the optimal investment strategy $\pi_l^*(t)$ for loan

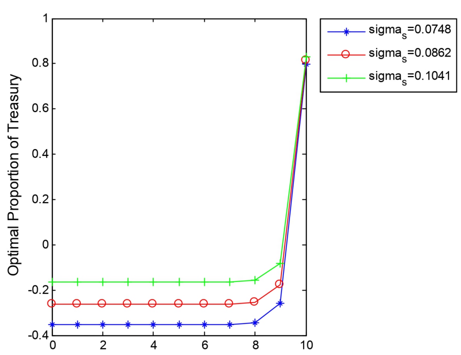

Figure 7: The effect of σ_s on the optimal investment strategy $\pi_0^*(t)$ for treasury

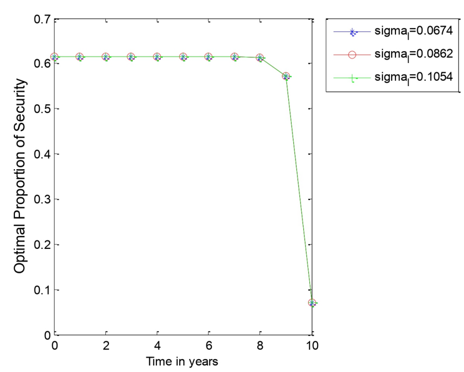

Figure 8: The effect of σ_l on the optimal investment strategy $\pi_s^*(t)$ for security

Figure 1 illustrates the trends of how the optimal proportion of the wealth invested in the three assets change with time. Figure 1 shows that the optimal proportion invested in the treasury is negative at the beginning of the investment horizon which indicates that the investor takes a short position in the treasury within this period. The investor short position in treasury within this period enables the investor invests more in the risky instruments but toward the end of the investment period, the investor invests more in the treasury and less in the risky assets to reach the optimal investment strategy. Therefore, the optimal investment strategy is to diversify the financial institution portfolio away from the risky assets toward the riskless treasury.

Now, $1-\beta$ which measures the relative risk aversion of the investor is the coefficient of risk aversion. It indicates that the larger the value of β, the smaller the relative risk aversion. Hence, the more aggressive the investor is and therefore the more the investor wishes to invest in the risky assets (security and loan). A numerical example that illustrates this relationship is given in Figure 2 to Figure 4. From the graphs, the optimal proportion of investment in the security and loan increases

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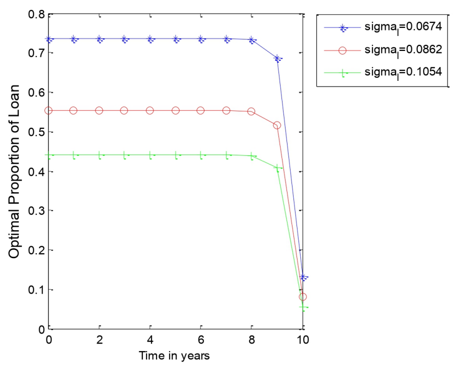

Figure 9: The effect of σ_l on the optimal investment strategy $\pi_l^*(t)$ for loan

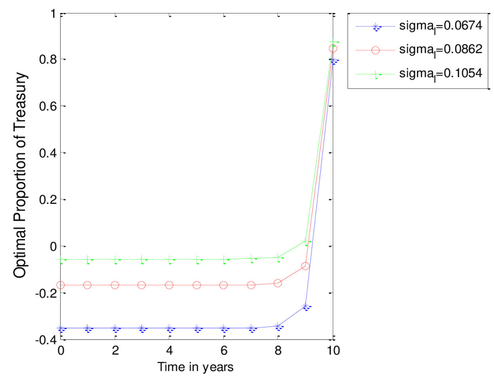

Figure 10: The effect of σ_l on the optimal Investment strategy $\pi_0^*(t)$ for treasury

as the parameter β increases as shown in Figure 2 and Figure 3 while the optimal proportion invested in treasury decreases as shown in Figure 4.

Figure 5 to 7 show the relationship between optimal investment strategy and the parameter σ_s. From Figure 6, we found that the optimal proportion invested in loan remains unchanged but the optimal proportion invested in the security decreases as σ_s increases as shown in Figure 5. This illustrates that the security volatility has no influence on the optimal investment in the loan. While the optimal proportion invested in the treasury increases as σ_s increases as shown in Figure 7. This illustrates the intuitive observation that if the optimal investment in loan remains unchanged and the optimal investment in the security decreases then the optimal investment in the treasury increases.

Figure 8 to 10 show the relationship between optimal investment strategy and the parameter σ_l. From Figure 8, we found that the optimal proportion invested in the security remains unchanged as σ_l increases while the optimal proportion invested in the loan decreases as σ_l increases as in Figure 9. But the optimal proportion invested in the treasury increases as the parameter σ_l increases as shown in Figure 10. This is also intuitive.

5 Conclusion

Allocating optimally the financial institution's resources among competing investments is very important. In this paper, we considered portfolio optimization problem of a financial institution where the interest rate is driven by Affine interest rate model. The volatilities of the security and loan are assumed to be constant. Here, the investor's objective is to maximize the expectation of CRRA utility of the terminal wealth. Under the portfolio optimization problem, the financial market consists of three assets namely; treasury, security and loan. We derived the optimal investment strategy for the case of CRRA utility function, obtained the explicit solution (of the resulted Hamilton – Jacobi – Bellman equation) for the optimal asset allocation problem and analyzed the behavior of the optimal portfolio via some numerical examples with interpretation of its economic meanings in the real market to verify the modification.

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