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New Subclasses of Univalent Functions Defined Using a Linear Combination of Generalised Sălăgean and Ruscheweyh Operators

Oluwayemi Matthew O. [1] And Fadipe-Joseph Olubunmi A.[2*],

1. Department of Mathematics, Landmark University, P.M.B. 1001, Omu-Aran, Nigeria
2. Department of Mathematics, University of Ilorin, P.M.B. 1515, Ilorin, Nigeria
*Corresponding author: oluwayemi.matthew@lmu.edu.ng

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Abstract

In the present paper, the authors investigated a subclass $T_\gamma D_{\lambda,\omega}(\alpha,\beta,\xi,\mu;p:n)$ of analytic functions in the open unit disk U. Using Ruscheweyh operator $R^n f_\gamma(z)$ and a generalized Sălăgean differential operator $D^n_{\lambda,\omega} f_\gamma(z)$ involving modified Sigmoid function, a class of normalized analytic function given by:

$$\Phi^n_{\lambda,\omega} f_\gamma(z) = \mu D^n_{\lambda,\omega} f_\gamma(z) + (1-\mu) R^n f_\gamma(z);\ \lambda,\omega \geq 0,\ \mu \in [0,1],\quad n \in N_0,\quad z \in U$$

was established.

Some geometric properties of the subclass $T_\gamma D_{\lambda,\omega}(\alpha,\beta,\xi,\mu;p:n)$ were investigated. The results extended and generalized some earlier results.

Keywords: Univalent functions, Sigmoid function, Differential operator, Fekete-Szego inequality, Starlikeness, Convexity, Close-to-convex funtions.
MSC2010: 30C45

1 Introduction and Preliminaries

Special functions are particular mathematical functions whose names and notations are due to their importance and applications in mathematical analysis, physics and other fields of science and engineering. Some special functions appear as solutions of differential equations. Symbolic computation where algorithms and software are required for manipulation of mathematical expressions make use of special functions. Activation functions are examples of special functions. Sigmoid function is the most popular of the three activation functions in the hardware implementation of Artificial Neural Network (ANN). In biologically inspired neural networks, the activation function is usually an abstraction representing the rate of action potential firing in the cell. Some of the sigmoid functions have been used as activation function of artificial neurons including the logistic and hyperbolic tangent functions. Sigmoid curves are also used as commutative distribution functions that go from 0 to 1 like the integrals of the logistics distribution, normal distribution, and student's

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t probability functions. The study of sigmoid function in geometric function theory is motivated by its significance in science, engineering and other fields of endeavours.
The study of analytic functions in complex analysis is significant because of their properties and applications but univalent functions have other interesting features aside being analytic which make its study of more interest. There are many subclasses of analytic and univalent functions in geometric function theory. A class T of functions with negative coefficients from second term was first introduced by Silverman [1] and has since then opened up a prolific line of research interest in that direction among function theorists.

Let A denotes the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \in C : |z| < 1\}$ and let

$$\gamma(s) = \frac{2}{1 + e^{-s}}; \quad s \geq 0 \tag{1.2}$$

be a modified sigmoid function then, $\gamma(s) = 1$ fo $s = 0$.

We denote by T the subclass of A consisting of functions $f(z) \in A$ which are analytic and univalent in U and of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0 \tag{1.3}$$

Hence, we have $f_\gamma(z) \in T_\gamma$ defined as

$$f_\gamma(z) = z - \sum_{k=2}^{\infty} \gamma(s) a_k z^k, \quad a_k \geq 0 \tag{1.4}$$

where $\gamma(s) = 1 + \frac{1}{2}s - \frac{1}{24}s^3 + \frac{1}{240}s^5 - \frac{17}{40320}s^7 + \ldots$ defined by (1.2). We also define identity function for T_γ as

$$e_\gamma(z) = z \tag{1.5}$$

We say that $f(z)$ is starlike in domain U if $f : U \to C$ is univalent and $f(U)$ is a starlike domain with respect to origin.
Then $f(z) \in A$ is said to be starlike of order ρ if it satisfies

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \rho$$

for some $(\rho(0 \leq \rho \leq 1))$ and for all $z \in U$. Also a univalent function $f(z) \in A$ is said to be convex of order ρ if and only if $zf'(z)$ is starlike of order ρ. In other words, if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \rho$$

for some $(\rho(0 \leq \rho \leq 1))$ and for all $z \in U$.

Further more, a univalent function $f(z) \in A$ is said to be close-to-convex of order ρ if

$$Re\left\{zf'(z)\right\} > \rho$$

for some $(\rho(0 \leq \rho \leq 1))$ and for all $z \in U$.

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2 Differential Operators

2.1 Sălăgean Differential Operator

Let $f(z) \in A$, the Sălăgean differential operator introduced in [2] denoted by $D^n f(z)$ is defined by

$$\begin{aligned} &D^0 f(z) = f(z) \\ &D^1 f(z) = z f'(z) \\ &\ \vdots \\ &D^n f(z) = z(D^{n-1} f(z))'. \end{aligned}$$

where $n \in N_0 = N \cup \{0\}$. If $f(z)$ is given by (1.1), then

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \tag{2.1}$$

Several other differential operators have been recently introduced to generalize (2.1) following the introduction of Sălăgean differential operator in [2]. Some of the operators include:
Al-Oboudi differential operator: Let $f(z) \in A$, in [3], Al-Oboudi defined a differential operator as follows

$$\begin{aligned} &D^0 f(z) = f(z) \\ &D_\lambda f(z) = D^1 f(z) = (1-\lambda) f(z) + \lambda z f'(z) = D_\lambda f(z), \quad \lambda \geq 0, \\ &\ \vdots \\ &D_\lambda^n f(z) = D_\lambda \left(D^{n-1} f(z) \right). \end{aligned}$$

Thus,

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} \{1 + (k-1)\lambda\}^n a_k z^k \quad \lambda \geq 0 \tag{2.2}$$

For $\lambda = 1$, (2.2) becomes (2.1).
Darus and Ibrahim in [4] defined a generalized differential operator

$$D_{\alpha,\beta,\lambda}^n f(z) = z + \sum_{k=2}^{\infty} \{\beta(k-1)(\lambda-\alpha) + 1\}^n a_k z^k \tag{2.3}$$

for $\alpha, \beta, \lambda \geq 0$, $k \geq 2$ and $n \in N_0 = N \cup \{0\}$.
Răducanu generalized Sălăgean and Al-Oboudi differential operators in [5] as follows

$$D_{\alpha,\lambda}^n f(z) = z + \sum_{k=2}^{\infty} [1 + (\alpha\lambda k + \alpha - \lambda)(k-1)]^n a_k z^k \tag{2.4}$$

Ramadan and Darus introduced a generalized differential operator in [6] as follows

$$D_{\alpha,\beta,\lambda,\delta}^n f(z) = z + \sum_{k=2}^{\infty} [(\lambda-\delta)(\beta-\alpha)(k-1) + 1]^n a_k z^k \tag{2.5}$$

for $\alpha, \beta, \delta \geq 0$, $\lambda > 0$, $\lambda > \delta$, $\beta > \alpha$, $k \geq 2$ and $n \in N_0 = N \cup \{0\}$.
In [7], Darus and Ibrahim introduced a generalized differential operator

$$\begin{aligned} &D^0 f(z) = f(z) \\ &D_{\alpha,\lambda}^1 f(z) = (\alpha-\lambda) f(z) + (\lambda-\alpha+1) z f'(z) \\ &\ \vdots \\ &D_{\alpha,\lambda}^n f(z) = D_{\alpha,\lambda}^1 \left(D^{n-1} f(z) \right). \end{aligned}$$

Thus,

$$D^n_{\alpha,\lambda} f(z) = z + \sum_{k=2}^{\infty} \{[(k-1)(\lambda-\alpha)+k]\}^n a_k z^k \quad \lambda \geq 0 \tag{2.6}$$

The operators are generalized form of some well-known differential operators such as Sălăgean operator (2.1) and Al-Oboudi operator (2.2) for example.

2.1.1 Differential Operator Involving modified Sigmoid Function

In [8], Fadipe-Joseph et. al introduced Sălăgean differential operator involving modified sigmoid function which is defined as follows
Let $f_\gamma(z) \in A_\gamma$, the Sălăgean differential operator denoted by $D^n f_\gamma(z)$ is defined by

$$\begin{aligned}
&D^0 f_\gamma(z) = f_\gamma(z)\\
&D^1 f_\gamma(z) = \gamma(s) z f'_\gamma(z)\\
&\ \vdots\\
&D^n f_\gamma(z) = D[D^{n-1} f_\gamma(z)] = \gamma(s) z [D^{n-1} f_\gamma(z)]'.
\end{aligned}$$

Hence,

$$D^n f_\gamma(z) = \gamma^n(s) z + \sum_{k=2}^{\infty} \gamma^m(s) k^n a_k z^k; \quad m = n+1 \tag{2.7}$$

for details, see [8].

2.1.2 New Differential Operator Involving modified Sigmoid Function

Let $f_\gamma(z) \in T_\gamma$, then from (2.6) and (2.7) we obtain a generalized differential operator involving modified sigmoid function as follows:

$$D^n_{\lambda,\omega} f_\gamma(z) = \gamma^n(s) z - \sum_{k=2}^{\infty} \gamma^{n+1}(s)[(k-1)(\lambda-\omega)+k]^n a_k z^k \tag{2.8}$$

for $\lambda, \omega \geq 0$. See [7] and [8] for detail.

2.1.3 Ruscheweyh Operator involving modified sigmoid function with $R^n : T_\gamma \to T_\gamma$, $n \in N_0 = N \cup \{0\}$

Let $f_\gamma(z) \in T_\gamma$, then the Ruscheweyh operator involving modified sigmoid function denoted by $R^n f_\gamma(z)$ is defined as

$$R^n f_\gamma(z) = z - \sum_{k=2}^{\infty} \gamma(s) B_k(n) a_k z^k \quad a_k \geq 0 \tag{2.9}$$

where,

$$\begin{aligned}
B_k(n) = B(n,k) = \begin{pmatrix} n+k-1 \\ n \end{pmatrix} &= \frac{(n+1)(n+2)\dots(n+k-1)}{(k-1)!} \\
&= \frac{(n+1)(n+2)\dots(n+k-1)}{(k-1)!} = \frac{(n+1)_{k-1}}{(1)_{k-1}}.
\end{aligned} \tag{2.10}$$

Hence, $B(0,k) = \begin{pmatrix} k-1 \\ 0 \end{pmatrix} = \dfrac{(1)_{k-1}}{(1)_{k-1}} = 1$. See [8] and [9] for detail.

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2.1.4 Combination of Generalised Sălăgean Differential Operator and Ruscheweyh Operator involving modified sigmoid function

We combine the generalised Sălăgean differential operator involving modified sigmoid function defined by (2.8) and the Ruscheweyh operator involving modified sigmoid function defined (2.9) above to obtain a certain operator defined as:

$$\Phi^n_{\lambda,\omega} f_\gamma(z) = \mu D^n_{\lambda,\omega} f_\gamma(z) + (1-\mu)R^n f_\gamma(z);\ \lambda \in [0,1],\ \mu \in [0,1]\ \ z \in U. \tag{2.11}$$

$$\Phi^n_{\lambda,\omega} f_\gamma(z) = \mu\gamma^n(s)z - \sum_{k=2}^{\infty} \mu\gamma^{n+1}(s)[(k-1)(\lambda-\omega)+k]^n a_k z^k + (1-\mu)z + \sum_{k=2}^{\infty}(1-\mu)\gamma(s)B_k(n)a_k z^k$$

$$= [\mu\gamma^n(s) - \mu + 1]z - \sum_{k=2}^{\infty} \gamma(s)\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n + (1-\mu)B_k(n)\right\} a_k z^k.$$

Subclasses of univalent functions in geometric function theory are characterized by simple analytic inequalities and play significant role in the study of univalent functions. Several authors such as [12], [4], [13], [7], [6], [5], [14], [15], [16], [17], [18], [19] and [20] have successfully defined and investigated various subclasses of univalent functions. In particular, Joshi and Sangle in [10] introduced and studied subclass $TD_\lambda(\alpha,\beta,\xi,\mu;n)$ of univalent functions by using Al-Oboudi operator as a generalised Sălăgean differential operator in the unit disk $U = \{z : |z| < 1\}$. This was motivated by the work of Joshi and Sangle [10]. Using differential operator defined in (2.11), a class $T_\gamma D_{\lambda,\omega}(\alpha,\beta,\xi,\mu;p:n)$ of univalent functions which extends and generalizes the class earlier studied in [10] was established.

2.2 Definition:

A function $f_\gamma(z) \in T_\gamma$ defined by (1.4) is in the class $T_\gamma D_{\lambda,\omega}(\alpha,\beta,\xi,\mu;p:n)$ if

$$\left| \frac{[\Phi^n_{\lambda,\omega} f_\gamma(z)]' - [\mu\gamma^n(s) - \mu + 1]}{p\xi\left[(\Phi^n_{\lambda,\omega} f_\gamma(z))' - \alpha\right] - \left[(\Phi^n_{\lambda,\omega} f_\gamma(z))' - (\mu\gamma^n(s) - \mu + 1)\right]} \right| < \beta$$

where $0 \le \alpha < \frac{1}{2}\xi,\ 0 < \beta \le 1,\ \frac{1}{2} \le \xi \le 1,\ \mu \in [0,1]$ *and* $p, n \in N_0 = N \cup \{0\}$; $n \ge 0$ *and* $p \ge 2\ \ z \in U$.

3 Main Results

In this section we state and prove the main results of this paper.
We begin by proving the necessary and sufficient condition for a function to belong to the class $T_\gamma D_{\lambda,\omega}(\alpha,\beta,\xi,\mu;p:n)$.

Theorem 3.1
If a function $f_\gamma(z)$ belongs to the class $T_\gamma D_{\lambda,\omega}(\alpha,\beta,\xi,\mu;p:n)$, then

$$\sum_{k=2}^{\infty} k\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n + (1-\mu)B_k(n)\right\} a_k$$
$$\le p\xi\beta[\mu\gamma^n(s) - \mu + 1 - \alpha].$$

Proof:
Suppose $f_\gamma(z)$ belongs to the class $T_\gamma D_{\lambda,\omega}(\alpha,\beta,\xi,\mu;p:n)$, by equation (2.11) and definition 2.2, we have that

$$\left| -\sum_{k=2}^{\infty} k\gamma(s)\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}a_k z^{k-1}\right|$$

$$\leq \beta\left|p\xi[\mu\gamma^n(s)-\mu+1-\alpha]-\sum_{k=2}^{\infty} k\gamma(s)(1-p\xi)\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}a_k z^{k-1}\right|$$

$|z| \leq r$ *and as* $r \to 1^+$, then

$$\sum_{k=2}^{\infty} k\gamma(s)\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}a_k$$

$$\leq \beta p\xi[\mu\gamma^n(s)-\mu+1-\alpha]+\sum_{k=2}^{\infty}\beta k\gamma(s)(1-p\xi)\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}a_k$$

$$\Rightarrow \quad \sum_{k=2}^{\infty} k\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}a_k$$

$$\leq p\xi\beta[\mu\gamma^n(s)-\mu+1-\alpha].$$

Hence,

$$\sum_{k=2}^{\infty} a_k \leq \frac{p\xi\beta[\mu\gamma^n(s)-\mu+1-\alpha]}{k\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}} \tag{3.1}$$

The result is sharp for

$$f(z) = z - \frac{p\xi\beta[\mu\gamma^n(s)-\mu+1-\alpha]}{k\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}}z^k; \quad k \geq 2.$$

Corollary 3.1
Let a function $f_\gamma(z)$ belongs to the class $T_1 D_{\lambda,\omega}(\alpha,\beta,\xi,\mu;p:n)$ then.

$$\sum_{k=2}^{\infty} k[1+\sigma(p\xi-1)]\left\{\mu[(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}a_k \leq p\xi\beta(1-\alpha).$$

Corollary 3.2
Let a function $f_\gamma(z)$ belongs to the class $T_1 D_{\lambda,1}(\alpha,\beta,\xi,\mu;p:n)$ then.

$$\sum_{k=2}^{\infty} k[1+\beta(p\xi-1)]\left\{\mu[(k-1)(\lambda-1)+k]^n+(1-\mu)B_k(n)\right\}a_k \leq p\xi\beta(1-\alpha).$$

Corollary 3.3
Let a function $f_\gamma(z)$ belongs to the class $T_1 D_{\lambda,1}(\alpha,\beta,\xi,1;p:n)$ then.

$$\sum_{k=2}^{\infty} k[1+\beta(p\xi-1)]\left\{[(k-1)(\lambda-1)+k]^n\right\}a_k \leq p\xi\beta(1-\alpha).$$

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Corollary 3.4
Let a function $f_\gamma(z)$ to the class $T_1D_{\lambda,0}(\alpha,\beta,\xi,1;2:n)$ then.

$$\sum_{k=2}^{\infty} k[1+\beta(2\xi-1)]\left\{[(k-1)\lambda+1]^n\right\}a_k \leq 2\xi\beta(1-\alpha).$$

The result is sharp for

$$f(z)=z-\frac{2\xi\beta(1-\alpha)}{k[1+\beta(2\xi-1)]\left\{[(k-1)\lambda+1]^n\right\}}z^k.$$

Remark: Class $T_1D_{\lambda,1}(\alpha,\beta,\xi,1;2:n) \equiv TD_\lambda(\alpha,\beta,\xi;n)$ studied in [10]. Thus,

$$TD_\lambda(\alpha,\beta,\xi;n) \subset T_\gamma D_{\lambda,\omega}(\alpha,\beta,\xi,\mu;p:n).$$

Theorem 3.2
Let $T_\gamma D_{\lambda,\omega}(\alpha,\beta,\xi,\mu;p:n)$, then

$$r-r^2\frac{p\xi\beta[\mu\gamma^n(s)-\mu+1-\alpha]}{2[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(\lambda-\omega)+2]^n+(1-\mu)B_2(n)\right\}} \leq |f_\gamma(z)|$$

$$\leq r+r^2\frac{p\xi\beta[\mu\gamma^n(s)-\mu+1-\alpha]}{2[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(\lambda-\omega)+2]^n+(1-\mu)B_2(n)\right\}}.$$

Proof:
By Theorem 3.1, for any function $f \in T_\gamma D_{\lambda,\omega}(\alpha,\beta,\xi,\mu;p:n)$, we have that

$$a_k \leq \sum_{k=2}^{\infty}\frac{p\xi\beta[\mu\gamma^n(s)-\mu+1-\alpha]}{k\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}} \quad (k=2,3,\ldots)$$

and

$$|f_\gamma(z)| \geq |z|-\sum_{k=2}^{\infty}\gamma a_k|z|^k \geq |z|-|z|^2\sum_{k=2}^{\infty}\gamma a_k$$

$$\geq |z|-|z|^2\sum_{k=2}^{\infty}a_k$$

$$|f_\gamma(z)| \geq r-r^2\frac{p\xi\beta[\mu\gamma^n(s)-\mu+1-\alpha]}{2[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(\lambda-\omega)+2]^n+(1-\mu)B_2(n)\right\}} \quad \textit{where} \quad |z|=r, \textit{ and } k=2. \tag{3.2}$$

Similarly,

$$|f_\gamma(z)| \leq |z|+\sum_{k=2}^{\infty}\gamma(s)a_k|z|^k \leq |z|+|z|^2\gamma(s)a_2$$

$$\Rightarrow |f(z)| \leq z+\gamma|z|^2\sum_{k=2}^{\infty}\gamma(s)a_k, \quad k \geq 2$$

$$|f_\gamma(z)| \leq r+\gamma r^2\frac{p\xi\beta[\mu\gamma^n(s)-\mu+1-\alpha]}{2\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(\lambda-\omega)+2]^n+(1-\mu)B_2(n)\right\}} \quad \textit{for} \quad |z|=r, \textit{ and } k=2. \tag{3.3}$$

From (3.2) and (3.3) we have

$$r - r^2 \frac{p\xi\beta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(\lambda-\omega)+2]^n + (1-\mu)B_2(n)\right\}} \leq |f_\gamma(z)|$$

$$\leq r + r^2 \frac{p\xi\beta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(\lambda-\omega)+2]^n + (1-\mu)B_2(n)\right\}}.$$

Theorem 3.3
Let $T_\gamma D_{\lambda,\omega}(\alpha,\beta,\xi,\mu;p:n)$, then

$$1 - r \frac{p\xi\beta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(\lambda-\omega)+2]^n + (1-\mu)B_2(n)\right\}} \leq |f'_\gamma(z)|$$

$$\leq 1 + r \frac{p\xi\beta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(\lambda-\omega)+2]^n + (1-\mu)B_2(n)\right\}}.$$

Proof:
Let $f_\gamma \in T_\gamma D_{\lambda,\omega}(\alpha,\beta,\xi,\mu;p:n)$, then, by Theorem 3.1, if $f \in T_\gamma D_{\lambda,\omega}(\alpha,\beta,\xi,\mu;p:n)$, we have that

$$\sum_{k=2}^{\infty} a_k \leq \frac{p\xi\beta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{k\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n + (1-\mu)B_k(n)\right\}} \quad (k = 2, 3, \ldots)$$

and thus,

$$a_2 \leq \frac{p\xi\beta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{2\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(\lambda-\omega)+2]^n + (1-\mu)B_2(n)\right\}} \quad (k = 2).$$

But

$$|f'_\gamma(z)| \geq 1 - \sum_{k=2}^{\infty} k\gamma(s)|a_k||z^{k-1}| \quad \texttt{and} \quad |f'_\gamma(z)| \leq 1 + 2\gamma(s)|z||a_2|.$$

So that

$$|f'_\gamma(z)| \geq 1 - k\gamma|z| \frac{p\xi\beta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{k\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n + (1-\mu)B_k(n)\right\}} \quad (for \;\; k \geq 2).$$

Hence,

$$|f'_\gamma(z)| \geq 1 - 2r\gamma \frac{p\xi\beta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{2\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(\lambda-\omega)+2]^n + (1-\mu)B_2(n)\right\}} \quad (for \;\; k = 2; \; |z| = r) \tag{3.4}$$

Also,

$$|f'_\gamma(z)| \leq 1 + 2r\gamma \frac{p\xi\beta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{2\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(\lambda-\omega)+2]^n + (1-\mu)B_2(n)\right\}} \quad (for \;\; k = 2; \; |z| = r) \tag{3.5}$$

Then, for $z \in U$, the equalities (3.4) and (3.5)

$$\Rightarrow \quad 1 - r\gamma \frac{p\xi\beta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(\lambda-\omega)+2]^n + (1-\mu)B_2(n)\right\}} \leq |f'_\gamma(z)|$$

$$\leq 1 + r\gamma \frac{p\xi\beta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(\lambda-\omega)+2]^n + (1-\mu)B_2(n)\right\}}$$

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which completes the proof.

Theorem 3.4

If a function $f_\gamma(z)$ defined by (1.4) belongs to the class $T_\gamma D_{\lambda,\omega}(\alpha,\beta,\xi,\mu;p:n)$.
Let

$$f_1(z)=z \ and \ f_\gamma(z)=z-\frac{p\xi\beta\gamma(s)[\mu\gamma^n(s)-\mu+1-\alpha]}{\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}}z^k,\ \ k\geq 2.$$

Then the function $f_\gamma \in T_\gamma D_{\lambda,\omega}(\alpha,\beta,\xi,\mu;p:n)$ if and only if it can be expressed in the form

$$f_\gamma(z)=\sum_{k=1}^{\infty}\mu_k f_k(z) \tag{3.6}$$

where $\mu_k\geq 0$ and $\sum_{k=1}^{\infty}\mu_k=1$.

Proof:

Let
$f_\gamma(z)=\sum_{k=1}^{\infty}\mu_k f_k(z)$; $\mu_k\geq 0$, $k=1,2,\ldots$ and $\sum_{k=1}^{\infty}\mu_k=1$.
Thus,

$$f_\gamma(z)=\sum_{k=1}^{\infty}\mu_k f_k=\mu_1 f_1(z)+\sum_{k=2}^{\infty}\mu_k f_k(z).$$

Thus,

$$f_\gamma(z)=\mu_1(z)+\sum_{k=2}^{\infty}\mu_k\left\{z-\frac{p\xi\beta[\mu\gamma^n(s)-\mu+1-\alpha]}{k\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}}z^k\right\}$$

$$=(\mu_1+\mu_2+\ldots)z-\sum_{k=2}^{\infty}\mu_k\frac{p\xi\beta[\mu\gamma^n(s)-\mu+1-\alpha]}{k\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}}z^k$$

$$\mu_1(z)+\mu_2 f_2(z)+\mu_3 f_3(z)+\ldots=\mu_1(z)+\sum_{k=2}^{\infty}\mu_k f_k$$

where $\mu_1+\mu_2+\ldots=\sum_{k=1}^{\infty}\mu_k=1$. Then,

$$f_\gamma(z)=z-\sum_{k=1}^{\infty}\mu_k\frac{p\xi\beta[\mu\gamma^n(s)-\mu+1-\alpha]}{k\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}}.$$

It thus follows that

$$\sum_{k=2}^{\infty}\mu_k\frac{p\xi\beta[\mu\gamma^n(s)-\mu+1-\alpha]}{k\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}}$$
$$\times\frac{k\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}}{p\xi\beta[\mu\gamma^n(s)-\mu+1-\alpha]}$$

$$\sum_{k=2}^{\infty}\mu_k=1-\mu_1\leq 1.$$

In otherwords,

$$f_\gamma(z)=\mu_1+\sum_{k=2}^{\infty}\mu_k=1\Rightarrow 1-\mu_1\leq 1.$$

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By Theorem 3.1 therefore, $f_\gamma \in T_\gamma D_{\lambda,\omega}(\alpha,\beta,\xi,\mu;p:n)$.
Conversely, if $T_\gamma D_{\lambda,\omega}(\alpha,\beta,\xi,\mu;p:n)$, then by Theorem 3.1,

$$a_k \leq \frac{p\xi\beta[\mu\gamma^n(s)-\mu+1-\alpha]}{k\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}} \quad (k \geq 2).$$

By setting

$$\mu_k \leq \frac{p\xi\beta[\mu\gamma^n(s)-\mu+1-\alpha]}{k\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}a_k}$$

and

$$\mu_1 = 1-\sum_{k=2}^{\infty}\mu_k.$$

So that

$$\mu_k = \frac{p\xi\beta[\mu\gamma^n(s)-\mu+1-\alpha]k\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}}{k\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}p\xi\beta[\mu\gamma^n(s)-\mu+1-\alpha]}.$$

We therefore notice that we can express f_k in terms of (3.6). Thus, $f_\gamma(z) = \sum_{k=1}^{\infty}\mu_k f_k$ which completes the proof.

3.1 Fekete-Szegǒ inequality for the class $T_\gamma D_{\lambda,\omega}(\alpha,\beta,\xi,\mu;p:n)$

In this section, the Fekete-Szegǒ inequality for functions $f_\gamma(z)$ belonging to the class $T_\gamma D_{\lambda,\omega}(\alpha,\beta,\xi,\mu;p:n)$ was established.

Theorem 3.5

If a function $f_\gamma(z) \in T_\gamma$ belongs to the class $T_\gamma D_{\lambda,\omega}(\alpha,\beta,\xi,\mu;p:n)$ and $\varphi \in R$. Then,

$$|a_3-\varphi a_2^2| \leq \frac{R[\Omega_2^2-\varphi R\Omega_1]}{\Omega_1\Omega_2^2}$$

Proof: From Equation 3.1,

$$a_2 = \frac{p\xi\beta[\mu\gamma^n(s)-\mu+1-\alpha]}{2\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(\lambda-\omega)+2]^n+(1-\mu)B_2(n)\right\}} \quad (k=2); \quad \textit{and}$$

$$a_3 = \frac{p\xi\beta[\mu\gamma^n(s)-\mu+1-\alpha]}{3\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)2(\lambda-\omega)+3]^n+(1-\mu)B_3(n)\right\}} \quad (k=3).$$

So that

$$\begin{aligned} a_3-\varphi a_2^2 &= \frac{p\xi\beta[\mu\gamma^n(s)-\mu+1-\alpha]}{3\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)2(\lambda-\omega)+3]^n+(1-\mu)B_3(n)\right\}} \\ &-\varphi\left\{\frac{p\xi\beta[\mu\gamma^n(s)-\mu+1-\alpha]}{2\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(\lambda-\omega)+2]^n+(1-\mu)B_2(n)\right\}}\right\}^2 \\ \Rightarrow |a_3-\varphi a_2^2| &\leq \left|\frac{R}{\Omega_2}-\frac{\varphi R^2}{\Omega_1^2}\right| = \left|\frac{R\Omega_1^2-\varphi R^2\Omega_2}{\Omega_2\Omega_1^2}\right| \end{aligned}$$

$$\textit{where } \Omega_1 = 2\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(\lambda-\omega)+2]^n+(1-\mu)B_2(n)\right\};$$
$$\Omega_2 = 3\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)2(\lambda-\omega)+3]^n+(1-\mu)B_3(n)\right\}; \textit{ and}$$
$$R = \left\{p\xi\beta[\mu\gamma^n(s)-\mu+1-\alpha]\right\}.$$

Therefore,

$$|a_3-\varphi {a_2}^2| \leq \frac{R[\Omega_2^2-\varphi R\Omega_1]}{\Omega_1\Omega_2^2}$$

which completes the proof.

3.2 Radius Properties for class $T_\gamma D_{\lambda,\omega}(\alpha,\beta,\xi,\mu;p:n)$

We now obtain the radii of starlikeness, convexity and close to convexity for the class Radii of close-to-convexity, starlikeness and convexity for $f_\gamma \in T_\gamma D_{\lambda,\omega}(\alpha,\beta,\xi,\mu;p:n)$ in this section as follows:

Theorem 3.6
Let the function $f_\gamma(z)$ be in the class $T_\gamma D_{\lambda,\omega}(\alpha,\beta,\xi,\mu;p:n)$; then $f_\gamma(z)$ is starlike of order ρ $(0 \le \rho < 1)$ in $|z| < r_1$, where

$$r_1 = inf_k \left\{ \frac{(1-\rho)k\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}}{p\xi\beta\gamma(s)[\mu\gamma^n(s)-\mu+1-\alpha](k-\rho)} \right\}^{\frac{1}{k-1}} \quad k \ge 2.$$

Proof: It suffices to show that $\left|\frac{zf'_\gamma(z)}{f_\gamma(z)} - 1\right| < 1-\rho$.
That is,

$$\left|\frac{zf'_\gamma(z)}{f_\gamma(z)} - 1\right| = \left|\frac{z-\sum_{k=2}^{\infty}\gamma(s)ka_kz^k - z + \sum_{k=2}^{\infty}\gamma(s)a_kz^k}{z-\sum_{k=2}^{\infty}\gamma(s)a_kz^k}\right|$$

$$\left|\frac{-\sum_{k=2}^{\infty}\gamma(s)(k-1)a_kz^{k-1}}{1-\sum_{k=2}^{\infty}\gamma(s)a_kz^{k-1}}\right| \le \frac{\sum_{k=2}^{\infty}\gamma(s)(k-1)a_k|z|^{k-1}}{(1-\sum_{k=2}^{\infty}\gamma(s)a_k|z|^{k-1})} < 1-\rho$$

It follows that

$$\sum_{k=2}^{\infty}\gamma(s)\frac{(k-\rho)|z|^{k-1}}{(1-\rho)} \le \frac{1}{a_k}$$

$$|z|^{k-1} \le \frac{(1-\rho)k\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}}{p\xi\beta\gamma(s)[\mu\gamma^n(s)-\mu+1-\alpha](k-\rho)}.$$

Equivalently,

$$|z| \le \left\{ \frac{(1-\rho)k\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}}{p\xi\beta\gamma(s)[\mu\gamma^n(s)-\mu+1-\alpha](k-\rho)} \right\}^{\frac{1}{k-1}}; \ |z| < r_1.$$

Thus,

$$r_1 = inf_k \left\{ \frac{(1-\rho)k\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}}{p\xi\beta\gamma(s)[\mu\gamma^n(s)-\mu+1-\alpha](k-\rho)} \right\}^{\frac{1}{k-1}} \quad k \ge 2$$

which completes the proof.

Theorem 3.7
Let the function $f_\gamma(z)$ be in the class $T_\gamma D_{\lambda,\omega}(\alpha,\beta,\xi,\mu;p:n)$, then $f_\gamma(z)$ is convex of order ρ $(0 \le \rho < 1)$ in $|z| < r_2$, where

$$r_2 = inf_k \left\{ \frac{(1-\rho)k\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}}{k(k-\rho)p\xi\beta\gamma(s)[\mu\gamma^n(s)-\mu+1-\alpha]} \right\}^{\frac{1}{k-1}} \quad k \ge 2 \tag{3.7}$$

Proof: It suffices to show that $\left|\frac{zf''_\gamma(z)}{f'_\gamma(z)}\right| < 1-\rho, \ |z| < r_2$.
Since

$$\left|\frac{zf''_\gamma(z)}{f'_\gamma(z)}\right| = \left|\frac{\sum_{k=2}^{\infty}\gamma(s)k(k-1)a_kz^{k-1}}{1-\sum_{k=2}^{\infty}k\gamma(s)a_kz^{k-1}}\right| \le \frac{\sum_{k=2}^{\infty}\gamma(s)k(k-1)a_k|z|^{k-1}}{1-\sum_{k=2}^{\infty}\gamma(s)ka_k|z|^{k-1}} < 1-\rho$$

To prove the Theorem, we must show that

$$\frac{\sum_{k=2}^{\infty}\gamma(s)k(k-1)a_k|z|^{k-1}}{1-\sum_{k=2}^{\infty}\gamma(s)ka_k|z|^{k-1}} < 1-\rho$$

$$\sum_{k=2}^{\infty}\gamma(s)k(k-\rho)a_k|z|^{k-1}\leq 1-\rho.$$

And by Theorem 3.1, we obtain

$$|z|^{k-1}\leq\frac{(1-\rho)k\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}}{k(k-\rho)p\xi\beta\gamma(s)[\mu\gamma^n(s)-\mu+1-\alpha]}$$

or

$$r_2=inf_k\left\{\frac{(1-\rho)k\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}}{k(k-\rho)p\xi\beta\gamma(s)[\mu\gamma^n(s)-\mu+1-\alpha]}\right\}^{\frac{1}{k-1}}$$

which completes the proof.

The result is sharp for the function $f_\gamma(z)$ given by

$$z-\frac{p\xi\beta\gamma(s)[\mu\gamma^n(s)-\mu+1-\alpha]}{\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}}z^k,\quad k\geq 2.$$

Theorem 3.8

Let the function $f_\gamma(z)$ be in the class $T_\gamma D_{\lambda,\omega}(\alpha,\beta,\xi,\mu;p:n)$. Then $f_\gamma(z)$ is closed-to-convex of order ρ $(0\leq\rho<1)$ in $|z|<r_3$, where

$$r_3=inf_k\left\{\frac{(1-\rho)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}}{p\xi\beta[\mu\gamma^n(s)-\mu+1-\alpha]}\right\}^{\frac{1}{k-1}}\ k\geq 2. \tag{3.8}$$

The result is sharp for the function $f_\gamma(z)$ given by

$$z-\frac{p\xi\beta\gamma(s)[\mu\gamma^n(s)-\mu+1-\alpha]}{\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}}z^k,\quad k\geq 2.$$

Proof: It suffices to show that $|f'_\gamma(z)-1|=1-\rho\ \ (0\leq\rho<1$ for $|z|<r_3$.

Thus,

$$|f'(z)-1|=\left|1-\sum_{k=2}^{\infty}k\gamma(s)a_kz^{k-1}-1\right|=\left|-\sum_{k=2}^{\infty}k\gamma(s)a_kz^{k-1}\right|\leq\left|\sum_{k=2}^{\infty}k\gamma(s)a_kz^{k-1}\right|.$$

$$Since\quad |f'_\gamma(z)-1|\leq\sum_{k=2}^{\infty}\gamma(s)ka_k|z^{k-1}|\leq 1-\rho\ \ if\ we\ divide\ bothsides\ by\ (1-\rho),\ then,$$

$$\sum_{k=2}^{\infty}\gamma(s)\left(\frac{k}{1-\rho}\right)a_k\left|z^{k-1}\right|\leq 1. \tag{3.9}$$

By coefficient estimates of $f_\gamma(z)\in T_\gamma D_{\lambda,\omega}(\alpha,\beta,\xi,\mu;p:n)$ given by Theorem 3.1 above, (3.8) holds if

$$\frac{k\gamma(s)|z|^{k-1}}{(1-\rho)}\leq\frac{k\gamma(s)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}}{p\xi\beta[\mu\gamma^n(s)-\mu+1-\alpha]}\ k\geq 2.$$

We find $(k-1)th$ root of both sides and multiply through by the inverse of $\frac{k\gamma}{(1-\rho)}$ so that

$$|z|\leq\left\{\frac{(1-\rho)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}}{p\xi\beta[\mu\gamma^n(s)-\mu+1-\alpha]}\right\}^{\frac{1}{k-1}}.$$

Hence,

$$r_3\leq\left\{\frac{(1-\rho)[1+\beta(p\xi-1)]\left\{\mu[\gamma^n(s)(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\right\}}{p\xi\beta[\mu\gamma^n(s)-\mu+1-\alpha]}\right\}^{\frac{1}{k-1}}.$$

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Conflict of Interest

The authors declare that there are no conflicts of interest regarding the publication of the paper.

References

[1] Silverman, H., Univalent functions with negative coefficients, *Proceedings of the American Mathematical Society* **51**, 109–116 (1975).

[2] Sălăgean, G.S., Subclasses of univalent functions, *Lecture Notes in Math, Springer-Verlag, Berlin* **1013**, 362–372 (1983).

[3] Al- Oboudi, F. M., On univalent functions defined by a generalized Sălăgean operator, *Int. J. Math. and Math. Sci, 27* **44**, 1429–1436 (2004).

[4] Darus, M. & Ibrahim, R. W., On subclasses for generalized operators of complex order, *Far East Journal of Mathematical Sciences* **33**, **3**, 299–308 (2009).

[5] Raducanu, D., On a subclass of univalent functions defined by a generalized differential operator, *Math. Reports 13* **63**, **2**, 197–203 (2011).

[6] Ramadan, S. F. & Darus, M., Univalence criteria for a family of integral operators defined by generalized differential operator, *Acta Universitatis Apulensis* **25**, 119–131 (2011).

[7] Darus, M. & Ibrahim, R. W., On new subclasses of analytic functions involving generalised differential and integral operators, *European Journal of Pure and Applied Mathematics (EJPAM)* **4**, 59–66 (2011).

[8] Fadipe-Joseph, O. A. , Moses, B.O. & Oluwayemi, M. O., Certain new classes of analytic functions defined by using sigmoid function, *Advances in Mathematics: Scientific Journal* **5**, 83–89 (2016).

[9] Ruscheweyh, S., New criteria for univalent functions, Proc. Amer. Math., *J. Indones Mathematics Society (MIHMI)* **49**, 109-115 (1975).

[10] Joshi, S.B. & Sangle, N.D., New subclass of univalent functions defined by using generalised Sălăgean operator, *J. Indones Mathematics Society (MIHMI)* **15**, 79–89 (2009).

[11] Sudharsan, T. V. & Vijaya, R., On certain subclasses of analytic and univalent functions based on an extension of Sălăgean operator, *Bonfring International journal of Data Mining* **2**, **3**, 1–5 (2012).

[12] Darwish, H.E., Certain subclasses of analytic functions with negative coefficients defined by generalised Sălăgean operator, *General Mathematics* **15**, **4**, 69–82 (2007).

[13] Najafzadeh, Sh. & Vijaya, R. Application of Sălăgean and Ruscheweyh operators on univalent functions with finitely many coefficients, *Fractional Calculus & Applied Analysis,* **13**, **5**, 1–5 (2010).

[14] Darus, M. & Faisal, I, Some subclasses of Analytic functions of complex order defined by new differential operator, *Tamkang Journal of Mathematics* **43**, **2**, 223–242. (2012).

[15] Keerthi, B. S. & Revathi, M., Certain new subclasses of analytic univalent functions in the unit disk, *Global Journal of Science Frontier Research Mathematics and Decision Sciences* **13**, (2013).

[16] Atshan, G. W., Khalaf, A. J. M. & Mahdi, M. M., On new subclass of univalent function with negative coefficient defined by hardamard product, *European Journal of Scientific Research* **119**, **3**, 462–472 (2014).

[17] Aldawish, I. & Darus, M., New subclass of analytic functions associated with the generalized hypergeometric functions, *Electronic Journal of Mathematical Analysis and Applications* **2**, **2**, 163–171 (2014).

[18] Murugusundaramoorthy, G. Certain subclasses of univalent functions associated with a unification of the Srivastava-Attiya and Cho-Saigo-Srivastava operators, *Novi Sad J. Math.* **45**, **2**, 59–76. (2015).

[19] Najafzadeh1, Sh., Ebadian, A. & Amini, E., Univalent functions with negative coefficients based on order of convolution consistence, *International Journal of Applied Mathematics* **28**, **5**, 579–591 (2015).

[20] Amourah, A. & Darus, M., Some properties of a new class of univalent functions involving a new generalized differential operator with negative coefficients, *Indian Journal of Science and Technology* **36**, **9**, 1–7 (2016).