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Fixed point theorems on A_p-metric spaces

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Abstract

The notion of A_p-metric spaces, which generalizes A-metric spaces, S-metric spaces and S_p-metric spaces is introduced in this paper. Analogues of some well known fixed point theorems are established and proved in this space with an application to the solution of a nonlinear integral equation. Our results generalize many known results in fixed point theory.

Keywords: A_p-metric spaces, A_p-Cauchy sequence, A_p-convergent and Fixed point.
MSC2010:47H10, 54H25.

1 Introduction

Metric space is an important tool in functional analysis, topology and nonlinear analysis. Its topological structure has attracted the attention of many mathematicians partly because of its usefulness in the fixed point theory. In recent years, diverse applications of fixed point theorems have challenged researchers to introduce different generalizations of metric spaces. These generalized spaces include 2-metric spaces, D-metric spaces, D^*-metric spaces, G-metric spaces, b-metric spaces, quasimetric spaces, G_b-metric spaces, complex valued G_b-metric spaces, S-metric spaces, S_b-metric spaces, complex valued S_b-metric spaces, A-metric spaces, γ-generalized quasi metric spaces and , most recently, S_p-metric spaces (see [1] to [16]).

Motivated by these generalizations, we present the notion of A_p-metric space which generalizes A-metric spaces, S-metric spaces and S_p-metric spaces. Some fixed point theorems are established and proved in this new space.

The following is the definition of S_p-metric spaces, a generalization of both S-metric spaces and S_b-metric spaces.
Definition 1.1 [1]. Let X be a non-empty set and $\overline{S} : X^3 \to \mathbb{R}^+$, a function with a strictly increasing continuous function, $\Omega : [0,\infty) \to [0,\infty)$ such that $\Omega(t) \geq t$ for all $t > 0$ and $\Omega(0) = 0$, satisfying the following properties:

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(i) $\overline{S}(x,y,z)=0 \qquad$ if and only if $x=y=z$;

(ii) $\overline{S}(x,y,z)\le\Omega(\overline{S}(x,x,a)+\overline{S}(y,y,a)+\overline{S}(z,z,a)) \qquad \forall a,x,y,z\in X$ (rectangle inequality).

Then $(X,\overline{S})$ is called an S_p-metric space.

Remark 1.2

(i) If $\Omega(z)=z$, S_p-metric space reduces to S-metric space.

(ii) If $\Omega(z)=bz$, S_p-metric space reduces to S_b-metric space.

In [2], Abass *et al.* introduced the notion of an A-metric space as follows:

Definition 1.3 [2]. A non-empty set X with a function $A:X^n\to[0,\infty)$ satisfying the following properties:

(i) $A(\phi_1,\phi_2,\phi_3,...,\phi_n)\ge 0$;

(ii) $A(\phi_1,\phi_2,\phi_3,...,\phi_n)=0$ if and only if $\phi_1=\phi_2=\phi_3=...=\phi_n$;

(iii) For $\phi_i,\varrho\in X, i=1,2,3,...,n$,

$$\begin{array}{rcl} A(\phi_1,\phi_2,\phi_3,...,\phi_n) & \le & A(\phi_1,\phi_1,\phi_1,...,(\phi_1)_{n-1},\varrho) \\ & + & A(\phi_2,\phi_2,\phi_2,...,(\phi_2)_{n-1},\varrho) \\ & + & A(\phi_3,\phi_3,\phi_3,...,(\phi_3)_{n-1},\varrho) \\ & + & A(\phi_4,\phi_4,\phi_4,...,(\phi_4)_{n-1},\varrho) \\ & . & \\ & . & \\ & . & \\ & + & A(\phi_n,\phi_n,\phi_n,...,(\phi_n)_{n-1},\varrho) \end{array}$$

is called an A-metric space.

Remark 1.4.

(i) A-metric space is an n-dimensional S-metric space in [2].

(ii) If $n=2$, A-metric space reduces to ordinary metric space in [3].

(iii) If $n=3$, A-metric space reduces to S-metric space in [4].

2 Main results

We introduce the following:
Definition 2.1. For a non-empty set X, let $\omega:[0,\infty)\to[0,\infty)$ be a strictly increasing continuous function such that $\omega^{-1}(z)\le z\le\omega(z)$ and $\omega(0)=0$ for all z. A mapping $A_p:X^n\to[0,\infty)$ satisfying the following properties:

(i) $A_p(\phi_1,\phi_2,\phi_3,...,\phi_n)\ge 0$;

(ii) $A_p(\phi_1,\phi_2,\phi_3,...,\phi_n)=0 \qquad$ if and only if $\phi_1=\phi_2=\phi_3=...=\phi_n$,

(iii) For $\phi_i, \varrho \in X, i = 1, 2, 3, ..., n$,

$$\begin{aligned} A_p(\phi_1, \phi_2, \phi_3, ..., \phi_n) &\leq \omega(A_p(\phi_1, \phi_1, \phi_1, ..., (\phi_1)_{n-1}, \varrho) \\ &+ A_p(\phi_2, \phi_2, \phi_2, ..., (\phi_2)_{n-1}, \varrho) \\ &+ A_p(\phi_3, \phi_3, \phi_3, ..., (\phi_3)_{n-1}, \varrho) \\ &+ A_p(\phi_4, \phi_4, \phi_4, ..., (\phi_4)_{n-1}, \varrho) \\ &. \\ &. \\ &. \\ &+ A_p(\phi_n, \phi_n, \phi_n, ..., (\phi_n)_{n-1}, \varrho)) \end{aligned}$$

is called an A_p-metric and (X, A_p) is a A_p-metric space.

Remark 2.2.

(i) A_p-metric space is an n-dimensional S_p-metric space. Every A-metric space is an A_p-metric space when $\omega(z) = z$ but the converse is not true.

(ii) If $n = 2$ and $\omega(z) = z$, A_p-metric space reduces to an ordinary metric space in [3].

(iii) If $n = 2$ and $\omega(z) = bz$, A_p-metric space reduces to b-metric space in [5].

(iv) If $n = 3$ and $\omega(z) = z$, A_p-metric space reduces to S-metric space in [4].

(v) If $n = 3$, A_p-metric space reduces to S_p-metric space in [2].

Example 2.3. Let $X = \mathbb{N} \bigcup \{0\}$ and

$$A_p(x_1, x_2, ..., x_n) = e^{\sum_{i=1}^{n-1} |x_1 - x_{i+1}|} - 1 \tag{2.1}$$

for all $x_i \in X, i = 1, 2, ..., n$ with $\omega(z) = e^z - 1$. Then (X, A_p) is an A_p-metric space.

Verification

(i)

$$A_p(x_1, x_2, ..., x_n) = e^{\sum_{i=1}^{n-1} |x_1 - x_{i+1}|} - 1 \geq 0$$

since exponential function is an increasing function.

(ii) If $x_1 = x_2 = x_3 = ... = x_n$,

$$\begin{aligned} A_p(x_1, x_2, ..., x_n) &= e^{\sum_{i=1}^{n-1} |x_1 - x_{i+1}|} - 1 \\ &= e^{|x_1 - x_2| + |x_1 - x_3| + |x_1 - x_4| + ... + |x_1 - x_n|} - 1 \\ &= e^{|0| + |0| + |0| + ... |0|} - 1 \\ &= e^0 - 1 = 1 - 1 = 0 \end{aligned}$$

Conversely, if

$$A_p(x_1, x_2, ..., x_n) = e^{\sum_{i=1}^{n-1} |x_1 - x_{i+1}|} - 1 = 0,$$

then

$$lne^{\sum_{i=1}^{n-1} |x_1 - x_{i+1}|} = ln1$$

which implies

$$\sum_{i=1}^{n-1} |x_1 - x_{i+1}| = 0.$$

Hence, $x_1 = x_{i+1} \forall i$.

(iii) Clearly with $\omega(z) = e^z - 1$,

$$\begin{aligned} A_p(x_1, x_2, x_3, ..., x_n) &\leq \omega(A_p(x_1, x_1, x_1, ..., (x_1)_{n-1}, \varrho) \\ &+ A_p(x_2, x_2, x_2, ..., (x_2)_{n-1}, \varrho) \\ &+ A_p(x_3, x_3, x_3, ..., (x_3)_{n-1}, \varrho) \\ &+ A_p(x_4, x_4, x_4, ..., (x_4)_{n-1}, \varrho) \\ &. \\ &. \\ &. \\ &+ A_p(x_n, x_n, x_n, ..., (x_n)_{n-1}, \varrho)) \end{aligned}$$

for $x_i, \varrho \in X, i = 1, 2, 3, ..., n$.

Remark 2.4.

1. Example 2.3 is an A_p metric space but not a metric space because if $n = 2$ and $A_p = d$, where d is a metric on X, then

$$\begin{aligned} d(x_1, x_2) &= e^{|x_1 - x_2|} - 1 \\ &> e^{|x_1 - x_3|} - 1 + e^{|x_3 - x_2|} - 1 \end{aligned}$$

for some $x_1, x_2, x_3 \in X$. For instance, if $x_1 = 6$, $x_2 = 16$ and $x_3 = 10$, we obtain:

$$\begin{aligned} d(6, 16) &= e^{10} - 1 = 22026.5 - 1 = 22025.5 \\ &> e^4 - 1 + e^6 - 1 \\ &= 54.6 - 1 + 403.4 - 1 \\ &= 456 \end{aligned}$$

2. Example 2.3 is not necessarily an S-metric space because if $n = 3$ and $A_p = S$, then

$$\begin{aligned} S(x_1, x_2, x_3) &= e^{|x_1 - x_2| + |x_1 - x_3|} - 1 \\ &> e^{|x_1 - x_4|} - 1 \\ &+ e^{|x_2 - x_4|} - 1 \\ &+ e^{|x_3 - x_4|} - 1 \end{aligned}$$

for some $x_1, x_2, x_3, x_4 \in X$. For instance, if $x_1 = 6$, $x_2 = 9$, $x_3 = 16$ and $x_4 = 10$, we obtain:

$$\begin{aligned} S(6, 9, 16) &= e^{3+10} - 1 = e^{13} - 1 = 442412.4 \\ &> e^4 - 1 + e^1 - 1 + e^6 - 1 \\ &= 54.6 - 1 + 2.7 - 1 + 403.4 - 1 \\ &= 457.7 \end{aligned}$$

3. Example 2.3 is not necessarily an A-metric space because if $A_p = A$, as in A-metric space, then

$$\begin{array}{rcl} A(x_1, x_2, ..., x_n) & = & e^{|x_1 - x_2| + |x_1 - x_3| + ... + |x_1 - x_{n-1}|} - 1 \\ & > & e^{|x_1 - y|} - 1 \\ & + & e^{|x_2 - y|} - 1 \\ & + & e^{|x_3 - y|} - 1 \\ & . & \\ & . & \\ & . & \\ & + & e^{|x_n - y|} - 1 \end{array}$$

for some $x_i, y \in X, i = 1, 2, ..., n$.

Lemma 2.5. Let (X, A_p) be an A_p-metric space. Then for $u, v, w \in X$ and $n \in \mathbb{N}$,

1. $A_p(u, u, u, ..., v) = \omega(A_p(v, v, v, ..., u))$.
2. $A_p(u, u, u, ..., v) \leq \omega((n-1)A_p(u, u, u, ..., t) + A_p(v, v, v, ..., t))$.

Proof

1.

$$\begin{array}{rcl} A_p(u, u, u, ..., v) & \leq & \omega(A_p(u, u, u, ..., u) \\ & + & A_p(u, u, u, ..., u) \\ & + & A_p(u, u, u, ..., u) \\ & + & A_p(u, u, u, ..., u) \\ & . & \\ & . & \\ & . & \\ & + & A_p(v, v, v, ..., v, u)) \\ & = & \omega(A_p(v, v, v, ..., v, u)). \end{array}$$

Also,

$$\begin{array}{rcl} A_p(v, v, v, ..., u) & \leq & \omega(A_p(v, v, v, ..., v) \\ & + & A_p(v, v, v, ..., v) \\ & + & A_p(v, v, v, ..., v) \\ & + & A_p(v, v, v, ..., v) \\ & . & \\ & . & \\ & . & \\ & + & A_p(u, u, u, ..., u, v)) \\ & = & \omega(A_p(u, u, u, ..., u, v)). \end{array}$$

Therefore, $A_p(u, u, u, ..., v) = \omega(A_p(v, v, v, ..., u))$.

2.

$$\begin{aligned} A_p(u,u,u,...,v) &\leq \omega(A_p(u,u,u,...,t) \\ &+ A_p(u,u,u,...,t) \\ &+ A_p(u,u,u,...,t) \\ &. \\ &. \\ &. \\ &+ A_p(v,v,v,...,t)) \\ &= \omega((n-1)A_p(u,u,u,...,t) + A_p(v,v,v,...,t)). \end{aligned}$$

Definition 2.6. Let (X, A_p) be an A_p-metric space. A sequence β_n in X is said to be;

1. A_p-convergent to a point $\mu \in X$ if for any $\epsilon > 0$, there exists a positive integer $N_\circ$ such that, for all $n \geq N_\circ$, $A_p(\beta_n, \beta_n, \beta_n, ..., \mu) < \epsilon$.
2. A_p-Cauchy if for any $\epsilon > 0$, there exists a positive integer $N_\circ$ such that, for all $n, m \geq N_\circ$, $A_p(\beta_m, \beta_n, \beta_n, ..., \beta_n) < \epsilon$.

Definition 2.7. An A_p-metric space is said to be A_p-complete if every A_p-Cauchy sequence in it is A_p-convergent in it.
Theorem 2.8.
Let (X, A_p) be a complete A_p-metric space. Suppose that the mapping $T : X \to X$ satisfies the contraction condition

$$A_p(T\xi_1, T\xi_2, T\xi_3, ..., T\xi_n) \leq aA_p(\xi_1, \xi_2, \xi_3, ..., \xi_n) \forall \xi_i \in X, i = 1, 2, 3, ..., n \tag{2.2}$$

where $0 < a < 1$. Then T has a unique fixed point in X.

Proof:
Choose $\xi_\circ \in X$ and set $\xi_n = T^n\xi_\circ$, $n \geq 1$. We have

$$\begin{aligned} A_p(\xi_n, \xi_{n+1}, \xi_{n+1}, \xi_{n+1}, ..., \xi_{n+1}) &= A_p(T\xi_{n-1}, T\xi_n, T\xi_n, T\xi_n, ..., T\xi_n) \\ &\leq aA_p(\xi_{n-1}, \xi_n, \xi_n, \xi_n, ..., \xi_n) \\ &\leq a^2A_p(\xi_{n-2}, \xi_{n-1}, \xi_{n-1}, \xi_{n-1}, ..., \xi_{n-1}) \\ &. \\ &. \\ &. \\ &\leq a^nA_p(\xi_0, \xi_1, \xi_1, \xi_1, ..., \xi_1). \end{aligned}$$

Thus, for $m < n$,

$$\begin{aligned}
A_p(\xi_n, \xi_m, \xi_m, \xi_m, ..., \xi_m) &\leq \omega(A_p(\xi_n, \xi_n, \xi_n, ..., \xi_n, \xi_{n+1}) \\
&+ (n-1)A_p(\xi_m, \xi_m, \xi_m, \xi_m, ..., \xi_m, \xi_{n+1})) \\
&= A_p(\xi_n, \xi_{n+1}, \xi_{n+1}, \xi_{n+1}, ..., \xi_{n+1}) \\
&+ (n-1)A_p(\xi_m, \xi_{n+1}, \xi_{n+1}, \xi_{n+1}, ..., \xi_{n+1}) \\
&\leq a^n A_p(\xi_0, \xi_1, \xi_1, \xi_1, ..., \xi_1) \\
&+ \omega((n-1)^2 A_p(\xi_{n+1}, \xi_{n+1}, \xi_{n+1}, ..., \xi_{n+1}, \xi_{n+2}) \\
&+ (n-1)A_p(\xi_m, \xi_m, \xi_m, \xi_m, ..., \xi_m, \xi_{n+2})) \\
&= a^n A_p(\xi_0, \xi_1, \xi_1, \xi_1, ..., \xi_1) \\
&+ a^{n+1}(n-1)^2 A_p(\xi_0, \xi_1, \xi_1, \xi_1, ..., \xi_1) \\
&+ (n-1)A_p(\xi_m, \xi_{n+2}, \xi_{n+2}, \xi_{n+2}, ..., \xi_{n+2}) \\
&\leq a^n A_p(\xi_0, \xi_1, \xi_1, \xi_1, ..., \xi_1) \\
&+ a^{n+1}(n-1)^2 A_p(\xi_0, \xi_1, \xi_1, \xi_1, ..., \xi_1) \\
&+ a^{n+2}(n-1)^2 A_p(\xi_0, \xi_1, \xi_1, \xi_1, ..., \xi_1) \\
&+ (n-1)A_p(\xi_m, \xi_{n+3}, \xi_{n+3}, \xi_{n+3}, ..., \xi_{n+3}) \\
&. \\
&. \\
&. \\
&\leq a^n A_p(\xi_0, \xi_1, \xi_1, \xi_1, ..., \xi_1) \\
&+ a^{n+1}(n-1)^2 A_p(\xi_0, \xi_1, \xi_1, \xi_1, ..., \xi_1) \\
&+ a^{n+2}(n-1)^2 A_p(\xi_0, \xi_1, \xi_1, \xi_1, ..., \xi_1) \\
&. \\
&. \\
&. \\
&+ a^m(n-1)^2 A_p(\xi_0, \xi_1, \xi_1, \xi_1, ..., \xi_1) \\
&+ (n-1)A_p(\xi_m, \xi_{m+1}, \xi_{m+1}, ..., \xi_{m+1}) \\
&\leq \left[\frac{a^n \eta}{1-a} + a^m(n-1)\right] A_p(\xi_0, \xi_1, \xi_1, ..., \xi_1)
\end{aligned}$$

where $\eta = (1 + an^2 - 2an)$.
As $n, m \to \infty$, $A_p(\xi_n, \xi_m, \xi_m, \xi_m, ..., \xi_m) \to 0$.
Therefore, $\{\xi_n\}$ is an A_p-Cauchy sequence. By completeness, there exists $\lambda \in X$ such that ξ_n is A_p-convergent to λ.
Suppose $T\lambda \neq \lambda$,

$$A_p(\xi_n, T\lambda, T\lambda, ..., T\lambda) \leq aA_p(\xi_{n-1}, \lambda, \lambda, ..., \lambda). \tag{2.3}$$

Taking the limit, we obtain

$$A_p(\lambda, T\lambda, T\lambda, ..., T\lambda) \leq 0 \tag{2.4}$$

a contradiction. So, $T\lambda = \lambda$.
Suppose $\lambda_1 \neq \lambda_2$ is such that $T\lambda_1 = \lambda_1$ and $T\lambda_2 = \lambda_2$. Then

$$A_p(T\lambda_1, T\lambda_2, T\lambda_2, ..., T\lambda_2) \leq aA_p(\lambda_1, \lambda_2, \lambda_2, ..., \lambda_2). \tag{2.5}$$

Implying

$$A_p(\lambda_1, \lambda_2, \lambda_2, ..., \lambda_2) \leq 0. \tag{2.6}$$

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A contradiction. So, $\lambda_1 = \lambda_2$.
This shows the uniqueness of the fixed point. □
Remark 2.9. If $A_p(T\xi_1, T\xi_2, T\xi_3, ..., T\xi_n)$ is set as $d(T\xi_1, T\xi_2)$, Theorem 2.8 reduces to well known Banach Contraction Principle in metric spaces.

Theorem 2.10.
Let (X, A_p) be a complete A_p-metric space. Suppose that the mapping $T : X \to X$ satisfies the condition

$$\begin{aligned} A_p(T\xi_1, T\xi_2, ..., T\xi_n) \;\; \leq \;\; & a \max\{A_p(\xi_1, T\xi_1, T\xi_1, ..., T\xi_1), \\ & A_p(\xi_2, T\xi_2, T\xi_2, ..., T\xi_2), \\ & . \\ & . \\ & ., \\ & A_p(\xi_n, T\xi_n, T\xi_n, ..., T\xi_n)\}, \end{aligned}$$

$\forall \xi_i \in X, i = 1, 2, 3, ..., n$, where $0 < a < 1$. Then T has a unique fixed point in X.

Proof:
Choose $\xi_\circ \in X$ and set $\xi_n = T^n\xi_\circ$, $n \geq 1$. We have

$$\begin{aligned} A_p(\xi_n, \xi_{n+1}, \xi_{n+1}, \xi_{n+1}, ..., \xi_{n+1}) &= A_p(T\xi_{n-1}, T\xi_n, T\xi_n, T\xi_n, ..., T\xi_n) \\ &\leq a \max\{A_p(\xi_{n-1}, \xi_n, \xi_n, \xi_n, ..., \xi_n), \\ &\quad A_p(\xi_n, \xi_{n+1} + \xi_{n+1} + ... + \xi_{n+1})\} \end{aligned}$$

If the maximum is $A_p(\xi_n, \xi_{n+1} + \xi_{n+1} + ... + \xi_{n+1})$, then from (2.12) we obtain $A_p(\xi_n, \xi_{n+1} + \xi_{n+1} + ... + \xi_{n+1}) \leq aA_p(\xi_n, \xi_{n+1} + \xi_{n+1} + ... + \xi_{n+1})$ which is a contradiction.
Therefore, the maximum is $A_p(\xi_{n-1}, \xi_n, \xi_n, \xi_n, ..., \xi_n)$. Hence, we have

$$\begin{aligned} A_p(\xi_n, \xi_{n+1}, \xi_{n+1}, \xi_{n+1}, ..., \xi_{n+1}) &= A_p(T\xi_{n-1}, T\xi_n, T\xi_n, T\xi_n, ..., T\xi_n) \\ &\leq aA_p(\xi_{n-1}, \xi_n, \xi_n, \xi_n, ..., \xi_n) \\ &\leq a^2 A_p(\xi_{n-2}, \xi_{n-1}, \xi_{n-1}, \xi_{n-1}, ..., \xi_{n-1}) \\ & . \\ & . \\ & . \\ &\leq a^n A_p(\xi_0, \xi_1, \xi_1, \xi_1, ..., \xi_1). \end{aligned}$$

Thus, for $m < n$,

$$\begin{aligned}
A_p(\xi_n,\xi_m,\xi_m,\xi_m,...,\xi_m) &\leq \omega(A_p(\xi_n,\xi_n,\xi_n,...,\xi_n,\xi_{n+1}) \\
&+ (n-1)A_p(\xi_m,\xi_m,\xi_m,\xi_m,...,\xi_m,\xi_{n+1})) \\
&= A_p(\xi_n,\xi_{n+1},\xi_{n+1},\xi_{n+1},...,\xi_{n+1}) \\
&+ (n-1)A_p(\xi_m,\xi_{n+1},\xi_{n+1},\xi_{n+1},...,\xi_{n+1}) \\
&\leq a^n A_p(\xi_0,\xi_1,\xi_1,\xi_1,...,\xi_1) \\
&+ \omega((n-1)^2 A_p(\xi_{n+1},\xi_{n+1},\xi_{n+1},...,\xi_{n+1},\xi_{n+2}) \\
&+ (n-1)A_p(\xi_m,\xi_m,\xi_m,\xi_m,...,\xi_m,\xi_{n+2})) \\
&= a^n A_p(\xi_0,\xi_1,\xi_1,\xi_1,...,\xi_1) \\
&+ a^{n+1}(n-1)^2 A_p(\xi_0,\xi_1,\xi_1,\xi_1,...,\xi_1) \\
&+ (n-1)A_p(\xi_m,\xi_{n+2},\xi_{n+2},\xi_{n+2},...,\xi_{n+2}) \\
&\leq a^n A_p(\xi_0,\xi_1,\xi_1,\xi_1,...,\xi_1) \\
&+ a^{n+1}(n-1)^2 A_p(\xi_0,\xi_1,\xi_1,\xi_1,...,\xi_1) \\
&+ a^{n+2}(n-1)^2 A_p(\xi_0,\xi_1,\xi_1,\xi_1,...,\xi_1) \\
&+ (n-1)A_p(\xi_m,\xi_{n+3},\xi_{n+3},\xi_{n+3},...,\xi_{n+3}) \\
&. \\
&. \\
&. \\
&\leq a^n A_p(\xi_0,\xi_1,\xi_1,\xi_1,...,\xi_1) \\
&+ a^{n+1}(n-1)^2 A_p(\xi_0,\xi_1,\xi_1,\xi_1,...,\xi_1) \\
&+ a^{n+2}(n-1)^2 A_p(\xi_0,\xi_1,\xi_1,\xi_1,...,\xi_1) \\
&. \\
&. \\
&. \\
&+ a^m(n-1)^2 A_p(\xi_0,\xi_1,\xi_1,\xi_1,...,\xi_1) \\
&+ (n-1)A_p(\xi_m,\xi_{m+1},\xi_{m+1},...,\xi_{m+1}) \\
&\leq \left[\frac{a^n\eta}{1-a} + a^m(n-1)\right] A_p(\xi_0,\xi_1,\xi_1,...,\xi_1)
\end{aligned}$$

where $\eta = (1 + an^2 - 2an)$.
As $n, m \to \infty$, $A_p(\xi_n,\xi_m,\xi_m,\xi_m,...,\xi_m) \to 0$.
Therefore, $\{\xi_n\}$ is an A_p-Cauchy sequence. By completeness, there exists $\lambda \in X$ such that ξ_n is A_p-convergent to λ.
Suppose $T\lambda \neq \lambda$,

$$\begin{aligned}
A_p(\xi_n,T\lambda,T\lambda,...,T\lambda) &\leq a\max\{A_p(\xi_{n-1},\xi_n,\xi_n,...,\xi_n), \\
&\quad A_p(\lambda,T\lambda,T\lambda,...,T\lambda)\}.
\end{aligned}$$

Taking the limit, we obtain

$$A_p(\lambda,T\lambda,T\lambda,...,T\lambda) \leq 0 \tag{2.7}$$

A contradiction. So, $T\lambda = \lambda$.
Suppose $\lambda_1 \neq \lambda_2$ is such that $T\lambda_1 = \lambda_1$ and $T\lambda_2 = \lambda_2$. Then

$$\begin{aligned}
A_p(T\lambda_1,T\lambda_2,T\lambda_2,...,T\lambda_2) &\leq a\max\{A_p(\lambda_1,T\lambda_1,T\lambda_1,...,T\lambda_1), \\
&\quad A_p(\lambda_2,T\lambda_2,T\lambda_2,...,T\lambda_2)\}.
\end{aligned}$$

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Implying

$$A_p(\lambda_1, \lambda_2, \lambda_2, ..., \lambda_2) \leq 0. \tag{2.8}$$

A contradiction. So, $\lambda_1 = \lambda_2$.
This shows the uniqueness of the fixed point. □

Remark 2.11. If $A_p(T\xi_1, T\xi_2, T\xi_3, ..., T\xi_n)$ is set as $d(T\xi_1, T\xi_2)$ and $d(\xi_1, \xi_2)$ denotes the maximum, Theorem 2.10 reduces to well known Banach Contraction Principle in metric spaces.

3 An Application to the Solution of A Nonlinear Integral Equation

Consider the following nonlinear integral equation

$$\gamma(x) = \frac{1}{\lambda}\int_a^b g(x,s)H(x,s,s,...,\gamma(s))ds, \tag{3.1}$$

where $g : [a,b] \times [a,b] \to \mathbb{R}$, $H : [a,b]^n \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $\lambda \in \mathbb{R}$, $\lambda \neq 0$ is a given number.
Let X be the set of all real continuous functions $\gamma : [a,b] \to \mathbb{R}$ for which γ is continuous and $\gamma(x) \subset \mathbb{R}$ for each $x \in [a,b]$. endowed with the A_p metric

$$A_p(\gamma_1, \gamma_2, ..., \gamma_n) = e^{\sum_{i=1}^{n-1}|\gamma_1 - \gamma_{i+1}|} - 1$$

for all $\gamma_i \in X, i = 1, 2, ..., n$. Then (X, A_p) is a complete A_p-metric space. Consider the following conditions:

(i) There exists $\sigma > 0$ such that $|H(x,s,s,...,\rho_1) - H(x,s,s,...,\rho_2)| \leq \sigma|\rho_1 - \rho_2|$ for each $x, s \in [a,b]$ and each $\rho_1, \rho_2 \in \mathbb{R}$.

(ii) $\max\limits_{a\leq t\leq b} \int_a^b g(x,s)ds \leq 1$, for all $x, s \in [a,b]$.

Theorem 3.1. Suppose that Theorem 2.8 and the conditions $(i) - (ii)$ above holds. Then (3.1) has a unique solution.

Proof. Define a mapping $T : C[a,b] \to C[a,b]$ by

$$(T\gamma)(x) = \frac{1}{\lambda}\int_a^b g(x,s)H(x,s,s,...,\gamma(s))ds, \quad \forall x \in [a,b]. \tag{3.2}$$

Then the integral equation (3.1) is equivalent to the fixed point problem $\gamma = T\gamma$ where T is defined by (3.2).

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Now,

$$\begin{aligned}
A_p\left(T\gamma_1(s), T\gamma_2(s), T\gamma_3(s), ..., T\gamma_n(s)\right) &= \epsilon^{\sum_{i=1}^{n-1}|T\gamma_1(s)-T\gamma_{i+1}(s)|}-1\\
&= \epsilon^{\sum_{i=1}^{n-1}\left|\frac{1}{\lambda}\int_a^b g(x,s)H(x,s,s,...,\gamma_1(s))ds-\frac{1}{\lambda}\int_a^b g(x,s)H(x,s,s,...,\gamma_{i+1}(s))ds\right|}-1\\
&= \epsilon^{\frac{1}{|\lambda|}\int_a^b g(x,s)ds\sum_{i=1}^{n-1}|H(x,s,s,...,\gamma_1(s))-H(x,s,s,...,\gamma_{i+1}(s))|}-1\\
&\leq \epsilon^{\frac{\sigma}{|\lambda|}\sum_{i=1}^{n-1}|\gamma_1(s)-\gamma_{i+1}(s)|}-1\\
&= kA_p\left(\gamma_1(s), \gamma_2(s), \gamma_3(s), ..., \gamma_n(s)\right),
\end{aligned}$$

where $k = \epsilon^{\frac{\sigma}{|\lambda|}} \in [0, 1)$, $|\lambda| > \sigma$.
Therefore,

$$A_p\left(T\gamma_1(s), T\gamma_2(s), T\gamma_3(s), ..., T\gamma_n(s)\right) \leq kA_p\left(\gamma_1(s), \gamma_2(s), \gamma_2(s), ..., \gamma_2(s)\right)$$

and T has a unique fixed point which is the unique solution of the integral equation (3.1).

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Competing Financial Interests

The authors declare no competing financial interests.

References

[1] Mustafa, Z., Shahkoohi, R. J., Parvaneh, V., Kadelburg, Z. & Jaradat, M. M. M. Ordered S_p-metric spaces and some fixed point theorems for contractive mappings with application to periodic boundary value problems. *Fixed Point Theory and Applications* **2019**, 1–20 (2019).

[2] Abbas, M., Ali, B. & Suleiman, Y. I. Generalized coupled common fixed point results in partially ordered A-metric spaces. *Fixed Point Theory and Applications* **2015**, 1–24 (2015).

[3] Frechet, M. Sur queiques points du calcul fonctionnel. *Rendic. Circ. Mat. Palermo* **22**, 1–74 (1906).

[4] Sedghi, S., Shobe, N. & Aliouche, A. A generalization of fixed point theorem in S-metric spaces. *Mat. Vesn.* **64**, 258–266 (2012).

[5] Branciari, A. A fixed point theorem of Banach-Caccippoli type on a class of generalized metric spaces. *Publ. Math. Debrecen* **57**, 31–37 (2000).

[6] Adewale, O. K., Olaleru, J. O. & Akewe, H. Fixed point theorems on a quaternion valued G-metric spaces. *Communications in Nonlinear Analysis* **7**, 73–81 (2019).

[7] Dhage, B. Generalized metric space and mapping with fixed points. *Bulletin of the Calcutta Mathematical Society* **84**, 329–336 (1992).

[8] Gahler, S. 2-Metrische Raume und ihre topologische Struktur. *Mathematishe Nachrichten* **26**, 115–148 (1963).

[9] Mustafa, Z.& Sims, B. A new approach to generalized metric spaces. *J. Nonlinear Convex Analysis* **7(2)**, 289–297 (2006).

[10] Olaleru, J. O. Common fixed points of three self-mappings in cone metric spaces. *Applied Mathematics, E-Note* **11**, 41–49 (2010).

[11] Olaleru, J. O. & Samet, B. Some fixed point theorems in cone rectangular metric spaces. *J. Nigeria Mathematics Society* **33**, 145–158 (2014).

[12] Sedghi, S., Shobe, N. & Zhou, H. A common fixed point theorem in D*-metric spaces. *Fixed Point Theory and Applications* **2007**, Article ID 027906 (2007).

[13] Aghajani, A., Abbas, M. & Roshan, JR. Common fixed point of generalized weak contractive mappings in partially ordered G_b - metric spaces. *Filomat* **28(6)**, 1087–1101 (2014).

[14] Adewale, O. K., Olaleru, J. O., Olaoluwa, H. & Akewe, H. Fixed point theorems on a γ-generalized quasi-metric spaces. *Creative Mathematica and Informatics* **28**, 135–142 (2019).

[15] Adewale, O. K. & Osawaru, K. G-cone metric Spaces over Banach Algebras and Some Fixed Point Results. *International Journal of Mathematical Analysis and Optimization* **2019**, 546–557 (2019).

[16] Kakde, A., Biradar, S. S. & Hiremath, S. S. Solution of differential and integral equations using fixed point theory. *International Journal of Advanced Research in Computer Engineering and Technology* **3(5)**, 1656–1659 (2014).